# A MATHEMATICAL THEORY OF SPIRIT

BEING AN ATTEMPT TO EMPLOY CERTAIN MATHEMATICAL PRINCIPLES IN THE ELUCIDATION OF SOME METAPHYSICAL PROBLEMS

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Experimental Mensuration: an elementary Text-Book of Inductive Geometry. (Heinemann. 2s. 6d.)

# PREFACE

In the following attempt to elucidate certain metaphysical problems by the aid of mathematics we have constantly had in view the needs of the non-mathematical reader. The inclusion of discussions of certain elementary mathematical principles necessitated thereby we consider to be justified by the unfortunate fact that a large proportion of educated people have but a very slight acquaintance with the science of Mathematics. Of course, the chapters of this book dealing more or less exclusively with the discussion of such mathematical principles must not be regarded as in any way performing the functions of a mathematical text-book. In the first place: the discussion is always subordinated to the peculiar end in view of this book, and we have invariably checked the desire to discuss points of much mathematical importance and interest but which have no particular bearing on our final aims herein. In the second place: whenever we are concerned with some elementary mathematical principle or law, the validity of which is universally admitted by mathematicians, we give, as a rule, a geometrical or other illustration in preference to the formal proof of the law, partly for the reason that to many minds such illustrations are more interesting, and, in some cases, even more convincing, than a formal proof; and partly for the reason that the reader who desires the strictly logical and formal demonstration of any of these laws can always obtain such from any good mathematical text-book covering the same ground.

We send this book forth in the hope that, although its readers may criticise it as speculative, they will not deem it to be lacking in interest and utility.

We should here mention that part of the first chapter of this work has already appeared in *Morning Light*, a theological weekly publication, under the title "The Law of Correspondences as an Organon of Thought."

Our best thanks are due to Principal Sir Oliver Lodge, F.R.S., for permission to quote a rather long passage from his work on Mathematics, and to W. G. Llewellyn, Esq., and H. F. Trobridge, Esq., for kind assistance in reading the proofs, &c.

H. S. R.

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# A MATHEMATICAL THEORY OF SPIRIT

## CHAPTER I

### ON THE DOCTRINE OF CORRESPONDENCES

**Some Popular Ideas regarding the Spiritual.**

§ **1.** At the present day it appears that the world's thought is in a transition stage; everywhere do we find old theories rooted up, and the later ones which have been planted in their stead appear to be only just about to take root. The materialism which so powerfully swayed the philosophic and scientific thought of the nineteenth century has lost its power, and as yet no other mode of thought exerts a like sway, though it is clear that that mode which will ultimately gain the victory will be spiritual and idealistic rather than materialistic. It appears that what is needed is an organon of thought whereby we may mentally pass from the realm of matter to that of spirit, such an organon of thought being essential to the construction of a satisfactory system of spiritual philosophy. There seems, however, to be a tendency either so to materialise the idea of spirit that the distinction between matter and spirit vanishes, or else so to refine and transcendentalise the idea of spirit that one loses hold of it entirely. Both of these extremes of thought, neither of which can be regarded as satisfactory, are to be found, mixed in inextricable confusion, in the popular theology as well as the common superstitions, not only of the past, but also of the present time. As examples of such superstitions we may notice two curious old Breton customs. On the eve of All Souls' Day, it is thought by the superstitious Bretons that the dead come to visit the living. The people, therefore, gather in the churchyard and sing a plaint or "gwerz," afterwards returning to their homes to talk of their dead; and before retiring to bed, the housewife prepares a meal of pancakes, cream and cider wherewith the spirits may refresh themselves! The other custom is as follows: On St. John's night the "Tantad" or bonfire is lighted, around which the people assemble whilst the Litany for the Dead is recited. At the conclusion they march around in silence, and at the third round every one throws a flint into the embers. It is supposed that the spirits of the dead come to warm themselves amongst the remnants of the "Tantad," and whoever finds his stone turned over in the morning knows that he will die during the year![1] Similar ideas regarding the spirits of the dead, namely, that they need material food and drink, and the warmth of material fires, are very widespread; whence it follows, of course, that spirits are thought of as material beings, though of an attenuated and impalpable nature. Indeed, "matter deprived of its substance" describes the popular concept of a spirit as well as such a vague notion can be described. The fact that such an expression reduces to nothing and is practically meaningless is altogether in harmony with the common ideas on the subject.

**The Reality of the Spiritual.**

§ 2. In many minds the terms "spirit" and "spiritual" are associated with no other idea than that of a discarnate personality; but, in truth, a man is a spirit before that great experience which we call

1 See *Dealings with the Dead* (1898), translated by Mrs. A. E. Whitehead from LE BRAZ'S *La Légende de la Mort en Basse Bretagne.*

"death," equally as after. By the "spiritual" we understand the mental, the psychical, the ideal. Affection, thought, ideas, the things of the mind—these are spiritual, and of such does the spiritual world consist, the spiritual world within the mind of man, and the spiritual world of which the inner man is a part and in which he exists. Very generally, ideal existence appears to be regarded almost as non-existence—the things of the mental plane, such as thought and affection, are hardly admitted to be real. But as Descartes long ago made plain,[2] such things, the things of the spiritual world, are firmer realities than the things of the material world. We *can* doubt the existence of an external, material object corresponding to some sensation that we experience, we *cannot* doubt the existence of the sensation itself. Materialism would reduce all spiritual existences to terms of matter and motion; "the brain secretes consciousness," we are told, "as the liver secretes bile." But the mere fact that we know matter and motion only in terms of consciousness, that our knowledge of matter and motion is necessarily ideal, is in itself

[2] See DESCARTES' *Discourse on Method* and *Metaphysical Meditations*.

sufficient to render untenable the materialistic theory. Nor is that extreme form of idealism satisfactory which denies the existence of an external world, for such a world is demanded by the harmony of experience. "Both matter and spirit are true and real" as we have said elsewhere, "but on different planes of being."[3] The question arises, What is the relation or connection (if any) between them?—a question the answer to which would supply the needed organon of thought referred to above. Shall we accept the thorough-going Dualism which regards matter and spirit as conflicting elements, out of harmony with one another and ever opposed? Not so: for this theory can never give a satisfactory explanation of the Cosmos; for, by creating an irreconcilable antinomy between matter and spirit, so far from explaining the great Riddle of the Universe, it denies the possibility of an explanation.

**The Symbolism of the World of Matter.**

**§ 3.** There have, however, been some philosophers who, whilst regarding matter and spirit as quite distinct from one another, have held that the spiritual is prototype of the material, the material symbolical of the spiritual, and hence, that

[3] Redgrove: *Matter, Spirit and the Cosmos* (1910), § 86.

the worlds of spirit and of matter are closely related by the law of analogy. This view of the Cosmos is one of considerable antiquity. It is to be found in the writings of Plato, for whom this world was constructed after the pattern of the Archetypal, Ideal World. It is to be found in the body of literature known as the Kabalah—questionable with regard to authorship, but undoubtedly embodying the ancient traditions of Jewish philosophy—in the works of the Neo-Platonists, and in the writings of the sixteenth- and seventeenth-century theosophists and transcendentalists, Cornelius Agrippa for example, who drew mainly from these two sources; for the doctrine necessarily follows from the emanational theory of the origin of the Cosmos, which is the keynote of all these systems. All true mystics have caught a glimpse of it, and have more or less dimly realised and taught that Nature is a mighty parable speaking eloquently to the inward-seeing eye of the realities of the Spiritual Universe. And we may add that its truth is implied in all true poetry, for all true poetry partakes of the spirit of mysticism. Of Peter Bell we read that:

*A primrose by a river's brim*
*A yellow primrose was to him,*
*And it was nothing more.*[4]

But in a little primrose-tuft the poet sees;

*A lasting link in Nature's chain*
*From highest heaven let down!*[5]

In all these places, however, in Plato and Plotinus, in the *Zohar* and Agrippa of Nettesheim, in the mystics and poets, the doctrine of the symbolism of matter is but comparatively vaguely expressed, its significance only dimly realised.[6] It is in the Doctrine of Correspondences of the Swedish philosopher, Emanuel Swedenborg, that we find this view of the relation between matter and spirit first explicitly stated and clearly developed to its logical conclusions. Here, for the first time, is the principle of analogy between the material and the spiritual rendered definite and exact.

§ **4.** We find the germs of this doctrine, which was afterwards so fully developed by Swedenborg, in his earlier philosophical works. Thus, in the *Animal Kingdom*

[4] WORDSWORTH: *Peter Bell.*
[5] WORDSWORTH: *The Primrose of the Rock.*
[6] Modern philosophical inquiry has rendered Plato's ideal theory, for example, quite untenable.

**Swedenborg's Doctrine of Correspondences.**

he says, "In our Doctrine of Representations and Correspondences, we shall treat of both these symbolical and typical representations, and of the astonishing things which occur, I will not say in the living body only, but throughout nature, and which correspond so entirely to supreme and spiritual things, that one would swear that the physical world was purely symbolical of the spiritual world: inasmuch that if we chose to express any natural truth in physical and definite vocal terms, and to convert these terms only into the corresponding spiritual terms, we shall by this means elicit a spiritual truth or theological dogma, in place of the physical truth or precept; although no mortal would have predicted that anything of the kind would possibly arise by bare literal transposition, inasmuch as the one precept, considered separately from the other, appears to have absolutely no relation to it." [7] And,

[7] EMANUEL SWEDENBORG: *The Animal Kingdom, considered Anatomically, Physically, and Philosophically* (translated by J. J. G. Wilkinson, 1843), vol. i. p. 451 (footnote *u*). Commenting on this passage Emerson says: "The fact, thus explicitly stated, is implied in all poetry, in allegory, in fable, in the use of emblems, and in the structure of language. Plato knew of it, as is evident from his twice bisected line, in the sixth book of the Republic. Lord Bacon had found that

again, in the *Worship and Love of God* he writes, " . . . for such is the established correspondence, that by natural and moral truths, by means of the transpositions only of the expressions that signify natural things, we are introduced into spiritual truths, and *vice versa*. . . . For the sake of illustration, let one or two examples suffice, as first, *Light reveals the quality of its object, but the quality of the object appears according to the state of the light, wherefore the object is not always*

truth and nature differed only as seal and print ; and he instanced some physical propositions, with their translation into a moral or political sense. Behmen, and all the mystics, imply this law, in their dark riddle-writing. The poets, in as far as they are poets, use it ; but it is known to them only, as the magnet was known for ages, as a toy. Swedenborg first put the fact into a detached and scientific statement, because it was habitually present to him, and never not seen. It was involved . . . in the doctrine of identity and iteration, because the mental series exactly tallies with the material series. It required an insight that could rank things in order and series ; or, rather, it required such rightness of position, that the poles of the eye should coincide with the axis of the world. The earth had fed its mankind through five or six millenniums, and they had sciences, religions, philosophers ; and yet have failed to see the correspondence of meaning between every part and every other part. And, down to this hour, literature has no book in which the symbolism of things is scientifically opened. One would say, that, as soon as men had the first hint that every sensible object—animal, rock, river, air—nay, space and time, subsists *such as it appears;* as in the case of beauties, if they are objects viewed in varied light. Now if instead of light we take intelligence, the quality of the object of which is the truth of a thing; since intelligence is universally allowed to be spiritual light, this conclusion follows: *Intelligence discovers the truth of a thing, but the truth of a thing appears according to the state of the intelligence; wherefore that is not always true which is supposed to be true.* In like manner, if instead of intelligence, wisdom be called into corre-

not for itself, nor finally to a material end, but as a picture-language to tell another story of beings and duties, other science would be put by, and a science of such grand presage would absorb all faculties: that each man would ask of all objects what they mean: Why does the horizon hold me fast, with my joy and grief, in this centre? Why hear I the same sense from countless differing voices, and read one never quite expressed fact in endless picture-language? Yet, whether it be, that these things will not be intellectually learned, or that many centuries must elaborate and compose so rare and opulent a soul—there is no comet, rock-stratum, fossil, fish, quadruped, spider, or fungus that, for itself, does not interest more scholars and classifiers than the meaning and upshot of the frame of things." (*Representative Men.* "Swedenborg; or, the Mystic.") Emerson, we should note, was opposed to Swedenborg's theological views, hence he was not satisfied (as may be inferred from the above) with his working out in detail of the Doctrine of Correspondences. It would be beyond the limits of this work, however, to attempt a substantiation of these details.

spondence, the object of which is good; it then follows, *Wisdom manifests goodness, but the goodness of a thing appears according to the state of the wisdom; wherefore that is not always good that is believed to be good.*"[8] This, together with some twenty similar illustrations of correspondences, will also be found in his little work entitled *A Hieroglyphic Key to Natural and Spiritual Mysteries.*

**The Doctrine of Degrees.**

**§ 5.** We see, therefore, that if valid, the Doctrine of Correspondences constitutes an organon of thought whereby we can pass from the realm of matter to that of spirit, transmuting with a wonderful alchemy truths relating to the physical realm into those that are spiritual. If correspondence implied but an analogy of an arbitrary nature, a mere resemblance without any real connection, then the doctrine would be of little worth. But, as we shall see more fully later, correspondence implies and depends upon a very real connection, the causal relationship of spirit to matter. It will be of interest, therefore, to examine Swedenborg's Doctrine of Correspondences somewhat more fully; and, for a

[8] EMANUEL SWEDENBORG: *On the Worship and Love of God* (translated [by John Clowes], 1885) § 55 (footnote *s*), p. 107.

satisfactory discussion of this doctrine, it will be necessary, in the first place, to say something with regard to his Doctrine of Degrees.

Swedenborg distinguished between two sorts of degrees, which are to be found in everything that exists—"continuous degrees," and "discrete degrees." The first are such as gradually merge one into the other, as light merges into darkness and heat into cold. Discrete degrees, however, do not merge one into the other, but are related as end, cause and effect; and, although the end exists always in the cause, and both end and cause exist always in the effect, yet the three are perfectly distinct, or as Swedenborg puts it, "discrete." Taking the example of affection, thought and speech: the perfect distinction between the three is clear enough; affection does not merge into thought as heat merges into cold, nor does thought merge into speech as light merges into darkness; but for affection to exist as an end, it must exist in thought; and for thought to exist as a cause, it must exist and be manifested in speech or some other form of outward expression. Now, according to Swedenborg's philosophy, in God is the most universal End, or End of ends, and the

Spiritual Universe is related to the Physical Universe as cause is to effect. Therefore, since a "discrete" degree separates matter and spirit, they are perfectly distinct and do not merge one into the other. This distinction between matter and spirit is of a nature altogether different from that between two forms of matter, such, for example, as the solid and gaseous states. It resembles, rather, the distinction between matter and the ether of space, on the theory that matter is a sort of etheric phenomenon. Matter is, indeed, in the ultimate analysis, a spiritual phenomenon; or, as Carlyle so well puts it, "All visible things are emblems; what thou seest is not there on its own account; strictly taken it is not there at all: Matter exists only spiritually, and to represent some Idea, and *body* it forth." [9] And since spirit stands in the closer relationship to the One Only Real and Absolute Substance, that which is spiritual must be regarded as more real, more substantial than that which is material.

§ **6.** Now, between end, cause and effect there is an exact analogical relationship; for example, every affection has certain

[9] THOMAS CARLYLE: *Sartor Resartus*, Bk. I., ch. xi.

thoughts which are in harmony with it and immanent in which it exists, and for every thought there are spoken and written words which exactly represent it and by means of which it is outwardly expressed. It is this relation that is called "correspondence." Hence, as the spiritual universe is related to the physical universe as cause is to effect, everything of the one corresponds to everything of the other—everything physical has its correspondent on the spiritual plane, everything spiritual its correspondent on the physical plane; everything physical is a symbol, everything spiritual a prototype. We must not suppose, however, that this doctrine implies that there are no natural causes; but that such flow from spiritual causes. "There is nothing anywhere in the natural world which is in order," writes Swedenborg, "but derives its cause and origin from the spiritual world, that is, through the spiritual from the Divine." [10] We see therefore that correspondence implies a similarity of inner relationship between the universes of spirit and matter. For, since the material world exists as an effect flowing from the spiritual,

**The Analogical Relation between Cause and Effect.**

[10] EMANUEL SWEDENBORG: *Arcana Cœlestia* § 8211 (English translation, 1893, vol. x. p. 71).

constructed after a like plan, the things of the material world stand related to one another and to the whole of this world like as do the things of the spiritual world to one another and to the whole of that world. Those things that correspond are they which, existing on "discrete" planes of being, *i.e.*, planes which are related as cause is to effect, stand in the same relations to their respective planes of being.

**Use and Correspondence.**

§ **7.** Now, the uses or functions of anything depend upon the relations which it bears to the whole of its environment. From this it follows that those things that correspond have analogous uses or functions with reference to their respective planes of being; and this supplies us with a criterion for determining correspondences. As an example of this, let us consider the correspondence between foods and ideas. Just as various foods are necessary for the nourishment of the body, so are various knowledges or ideas necessary for the psychical nourishment of the mind. The body is built up of the foods eaten, but the body is not merely a conglomeration of foods: consisting of the various elements of food, the body is made up of these elements woven into a new form—

the body is a harmonious unity, a living organism. So also is the mind built up of ideas, and yet the mind is not merely a collection of ideas, for the ideas within the mind are also woven into a (more or less) harmonious unity—the mind is a living spiritual organism. Moreover, just as there are pseudo-foods that are injurious to the body, and bring discord amongst its elements, so also there are pseudo-knowledges that poison the mind, and produce discord amongst its elements. We see, therefore, that the functions of foods on the physical plane are analogous to those of knowledges or ideas on the spiritual plane, or, putting it otherwise, that the part played by foods in the economy of the natural world is similar to the part played by knowledges or ideas in the economy of the spiritual world. This is precisely what is meant when it is said that ideas and foods correspond.

**The Correspondence between the Soul and Body of Man.**

§ **8**. Not only can this exact analogy or symbolism be traced between the things of the spiritual and physical worlds in general, but also between the soul and body of man; for man considered spiritually is a microuranos—a Heaven in miniature—and with regard to his outward nature, as the old mystical philosophers taught, a microcosm or little world. Considering, for example, the two most important organs of the body, the heart and lungs: these correspond, according to Swedenborg, to the affectional and volitional, and the intellectual sides of man's spiritual nature respectively.[11] With regard to the first, the likeness in function is very evident, for just as the heart is the mainspring of man's physical life, so is his will or love the mainspring of his spiritual life. With regard to the second, we should notice that the function of the lungs is the cleansing and nourishing of the blood, and it is by the knowledge of what is good and true that man is cleansed from evil desires and spiritually vitalised.[12]

[11] See "Note on the Analysis of Consciousness" at the end of Chapter I.

[12] With reference to the correspondence between the soul and body of man we may, perhaps, quote the words of the Swedish philosopher himself: "Who is not aware," he says, "that affection and thought are spiritual, and therefore that all things of affection and thought are spiritual? Who is not aware that action and speech are natural, and therefore that all things of action and speech are natural? Who is not aware that affection and thought which are spiritual cause man to act and to speak? Who may not see on these grounds what the correspondence of spiritual things with natural is? Does not thought make the tongue speak, and affection together with thought make the

§ **9.** The assumption that appearance and reality are identical, that things are as they

body act ? There are two distinct things here. I can think and not speak, and I can will and not act. And it is known that the body does not think and does not will, but that thought falls into speech, and will into action. And does not affection shine forth out of the face, and present a type of itself there ? Every one knows this. Is not affection regarded by itself spiritual, and are not the changes of the face, the looks as they are called, natural ? Who might not conclude from this that there is a correspondence ; and hence that there is a correspondence of all things of the mind with all things of the body ? And because all things of the mind are referrible to affection and thought, or what is the same thing to the will and understanding, and all things of the body to the heart and the lungs, it might have been concluded that there is a correspondence of the will with the heart and of the understanding with the lungs. The cause why these things have not been known, although they might have been known, is that man has become so external that he has chosen to acknowledge nothing except the natural. This has been the delight of his love, and therefore it has been the delight of his understanding ; until it has become painful to him to elevate thought above the natural to anything spiritual separate from the natural. And so of his natural love and its delight he could not do otherwise than think that the spiritual is a purer natural, and that correspondence is a somewhat flowing in by continuity. Yea, the merely natural man is not able to think of anything separate from the natural ; such a thing to him is nothing. Another cause why these things have not been first seen and then known hitherto is, that all things of religion, all spiritual things, have been removed from the view of man by the dogma, prevailing in the whole Christian World, that the are perceived, although common enough, is nevertheless one that will not stand examination.[13] Not only does the appearance of objects depend upon the conditions under which they are observed, but also upon the sense-organs of the observer. Confining our attention

**Appearance and Reality.**

Theological things which are spiritual, which Councils and certain leaders have decreed, are to be blindly believed, because as they say they transcend the understanding. Whence some have supposed the spiritual to be like a bird which flies over the air in an ether to which the sight of the eye does not reach; when yet it is as a bird of paradise, which flies near the eye, and touches its pupil with its beautiful wings and longs to be seen. By the sight of the eye is understood the intellectual sight." (EMANUEL SWEDENBORG, *Angelic Wisdom concerning the Divine Love and concerning the Divine Wisdom*, §374, translated by J. J. Garth Wilkinson, 1885.) We should like to substitute "physical" for "natural" throughout the above passage, as more correctly rendering Swedenborg's meaning in modern terminology.

[13] The term "appearance" is here employed in its widest possible meaning, not as having reference only to visual sensations, but as implying the sum of all sensations which are in any way attributable to the external object in question. Of course, in a sense, an appearance is a reality. An appearance is real as an appearance, it is real as existing in the mind of the observer. The distinction in question, however, is one between objective and subjective reality, implying by these terms nothing more than reality which is true for all minds and that which exists for the individual mind alone, respectively.

for the moment to the consideration of visual percepts, as Swedenborg points out in the passage quoted in § 4, the appearance of objects depends upon the light in which they are observed. In this connection it is of interest to notice that researches in the domain of physical science have demonstrated the existence of ether-waves which have no effect upon our organs of sight. The colours of objects would appear quite different to a being whose eyes were sensitive to these ultra-violet rays, but unaffected (say) by red light rays, from what they appear to us. Indeed, one might go so far as to say that the appearances of objects depend equally upon the observer as upon the thing observed. The sense-organs of all men and women are very much the same, so that the world as it appears to each one of us does not greatly differ, but no two men or women see it exactly alike, because no two men or women see Nature through precisely the same spectacles.

**Spiritual Perception.**

§ **10.** Now, according to Swedenborg's philosophy, just as the outward man possesses physical senses whereby he may become informed of his material environment, so likewise the inward man possesses spiritual senses whereby he may become informed of his spiritual environment. During man's earth-life, however, save for somewhat rare exceptions—under which category the present writer would include all those phenomena called "telepathic"—these inward senses are opened only to the small spiritual world of his own mind. We can directly perceive our own thoughts; we can perceive the thoughts of others only in so far as they are expressed in a form which appeals to the outward senses. We can, as it were, see our own souls; we can see only the outward forms of other men and women. With that transition of the sphere of manifest activity and consciousness that we call "death," however, the wide spiritual world will open to our view, but simultaneously the physical world will be shut out, for just as the physical senses are affected only by that which is physical, so likewise are the senses of the spirit affected only by that which is itself spiritual. Further, the appearance of things spiritual depends upon the spiritual state of the percipient, just as the appearance of things material depends upon the state of the physical sense-organs. That this is actually so appears from such facts as that an unintelligent person cannot "see" the truth of some well-reasoned statement, whilst fallacies appeal to him as true; and that the evil-minded man "sees" good things as evil and undesirable, but evil things as good and greatly to be desired; whilst the man who is truly wise sees good things as good, evil as evil, true things as true, false as false.[14]

**Correspondence and Appearance.**

**§ 11.** It also follows from the Doctrine of Correspondences that those things that correspond are similar in appearance. The spirit-world, could we perceive it with our spiritual sight, would appear very much like this world of matter, its inhabitants like men and women here. In the spiritual world, however, the outward form of man

[14] In perception of the things of the physical world both the inward man and the outward man are involved. Hence, differences in our percepts of the same external object arise partly on account of differences in our sense-organs, &c., and partly on account of differences of a spiritual nature. In the above example of the dependence of percepts of physical things on the nature of the sense-organs, in which we supposed a being whose eyes were sensitive to ultra-violet light-rays, the effect of the spiritual state of the percipient is a negligible factor. In *Matter, Spirit, and the Cosmos*, § 46, where the subject of the relativity of spiritual perception is under discussion, a case is considered in which differences in perception due to differences in the sense-organs of the percipients are negligible in comparison with differences in perception due to differences in the spiritual states of the percipients.

corresponds exactly with his inward state, and the environment of each spirit is also in strict harmony with that spirit's inward state. There, the external is the exact expression of the internal, the outward world as perceived is the objectification of the subjective world within the mind. Such ideas as these regarding the nature and appearance of the spiritual world may appear to be highly speculative, but they do follow logically from the fundamental statement of the Law of Correspondences, and if we admit the validity of Swedenborg's seership, for which there is considerable evidence, they may be regarded as based upon experience. It is also of interest to note that they are in agreement with many statements said to be made by discarnate spirits through the mediumship of various "psychics"—testimony, it is true, which needs much careful and critical examination, and certainly should be received in no credulous spirit.[15]

The Doctrine of Correspondences states the constancy of the ratio between matter and spirit, a ratio which is inexpressible in

[15] For a critical discussion of Modern Spiritualism and Psychical Research, see the present writer's *Matter, Spirit, and the Cosmos* (Rider, 1910), Ch. II.

terms of experience, for spirit perceives only that which is spiritual, and the physical senses are affected only by the things of the physical world—a ratio, which, from a certain point of view, may be said to be unreal. As Louis Claude de Saint-Martin put it, "Matter is true for matter, and never for spirit." [16] We might add, Spirit is true for spirit, and never for matter. Between matter and spirit it is as if a great gulf were fixed, and it is this gulf which the Law of Correspondences claims to enable us mentally to bridge over.

. . . . .

**On Mathematical Language.**

§ 12. Without going to the extent of Prof. Max Müller's theory [17] and maintaining the absolute dependence of thought on words, it must, nevertheless, be confessed that language plays a very important part in the development of thought, and that one of the necessities of any system of philosophy is an adequate language, not only by means of which it may be expressed, but also by means of which it may be thought out. Now, there is no language so explicit, no

[16] A. E. Waite: *The Life of Louis Claude de Saint-Martin, the Unknown Philosopher* (1901), p. 185.

[17] See F. Max Müller: *Three Introductory Lectures on the Science of Thought* (1888).

language so univocal, no language so terse and yet so full of meaning, as the language of mathematics. Such is the nature of mathematical language that it may be said that no knowledge is worthy of being called scientific until it is expressed therein. The justice of this statement will become apparent if we compare the state of certain of the sciences (*e.g.*, Chemistry) to-day with their condition but a generation ago, and consider the fact that the enormous advances which have been made are due in large measure to the application to these sciences of mathematical principles and methods.

The thought occurs to us, Might not mathematics be advantageously employed in the domain of Metaphysics? How great an advantage it would be if, indeed, the relation between matter and spirit could be represented mathematically, if a *mathematical* organon of thought were forthcoming whereby we could mentally pass from the physical realm to the spiritual, and deduce, thereby, spiritual principles from natural laws!

**§ 13.** Now, such, in brief, are the objects of the present work; herein we shall attempt to formulate a mathematical organon of thought whereby we can mentally pass from the realm of the physical to that of the spiritual, and by means of this organon of thought, to arrive at and establish certain spiritual or metaphysical principles. Whether or not we are successful in these attempts must be left to the reader's decision after he has perused the volume. We have discussed Swedenborg's Doctrine of Correspondences at some length above (though we have by no means exhausted all of its aspects), for the reason that the results obtained in this endeavour to construct a mathematical theory of the spiritual are in remarkable agreement with the several implicates of this doctrine. To those, therefore, who regard Swedenborg's Law of Correspondences as valid, we offer the present volume as an attempt to cast this law into a mathematical mould, whereby it may gain in precision and usefulness. At the same time, however, it will be our aim, herein, to establish our theory by an entirely different line of argument, so that the reader is not asked to accept Swedenborg's doctrine before proceeding with the book. If our arguments are sound, however, they may fairly be regarded as affording some evidence of the validity of this doctrine.

**Mathematics and the Philosophy of the Spiritual.**

## Note on the Analysis of Consciousness

Swedenborg's analysis of consciousness is dual, as distinguished from the more modern, triple analysis into Cognition ( =knowing), Feeling-attitude, and Conation ( =willing). It is interesting to note, however, that, as Prof. Stout remarks, "Of late there has again arisen a tendency to fall back upon a dual division, bringing Feeling-attitude and Conation under the same head." "It is clear," proceeds this eminent psychologist, "that they are more closely akin to each other than either of them is to Cognition. It also seems clear that there is as fundamental a distinction between the bare thought of an object and the affirmation or denial of its reality as there is between Feeling-attitude and Conation. The best plan is to adopt a most comprehensive dual division into 'Cognition' on the one hand and 'Interest' on the other. Cognition may then be subdivided under the heads, Simple Apprehension and Judgment; and Interest may be subdivided under the heads Conation and Feeling-attitude." (G. F. Stout : *The Groundwork of Psychology*, 1905, pp. 18–19.) It must be understood, of course, that all such analyses are purely ideal. Cognition, Feeling-attitude, and Conation cannot be actually separated, for all three exist together (though in varying degrees of intensity) in every moment of consciousness. But no more can the organs of the living body be separated, for with their removal they cease to be living organs and become merely inert masses of matter.

# CHAPTER II

## ON INCOMMENSURABLE QUANTITIES

**Powers and Roots.**

§ **14.** THE continued product of a number with itself, that is to say, the product obtained by continually multiplying a number by itself, is called a *power* of that number. Thus, $3 \times 3$ is called the *second power* of 3; $3 \times 3 \times 3$ is called the *third power* of 3; $3 \times 3 \times 3 \times 3$ is called the *fourth power* of 3, &c. The second and third powers of a number are termed also the *square* and *cube* of that number, respectively. The second power or square of 3 is written $3^2$; the third power or cube, $3^3$; the fourth power, $3^4$; and so on. The small figure (written above and to the right of the number) which expresses how often that number is repeated to form the power, is called its *index* or *exponent*. In general—

The second power or square of $x$ is written $x^2$ and equals $x \times x$;

The third power or cube of $x$ is written $x^3$ and equals $x \times x \times x$;

The fourth power of $x$ is written $x^4$ and equals $x \times x \times x \times x$, and so on.

The process of determining the value of a power is called *Involution*.

The number which gives rise to a power is called its *root*. Thus 3 is said to be the *square root* of 9, since 9 is the square of 3; 2 is said to be the *cube root* of 8, since 8 is the cube of 2, &c. The symbol $\surd$ is used to denote the root of a number; thus, the square root of 9 is written $\sqrt[2]{9}$, or more simply $\sqrt{9}$, the 2 in $\sqrt[2]{\ }$ generally being omitted; the cube root of 8 is written $\sqrt[3]{8}$; similarly, a fourth root is indicated by the sign $\sqrt[4]{\ }$, and so on. The symbol $\surd$ is called the *radical sign*. The process of determining the value of a root is called *Evolution*; hence involution and evolution are inverse processes, *i.e.*, one undoes the effect of the other.

**The Determination of Square Roots.**

**§ 15.** Methods for the extraction of square and certain other roots will be found in any text-book of Arithmetic. Here, we shall be concerned only with square roots. Now, in the case of certain numbers no difficulty whatever is experienced in determining exactly their square roots. Thus, for example, $\sqrt{4} = 2$ exactly, $\sqrt{9} = 3$ exactly, &c.,

or, as an example of a larger number, it is found that $\sqrt{7569}=87$ exactly, &c. Such numbers, however, are the exception rather than the rule; it is found that the exact square roots of the vast majority of whole numbers (all whole numbers, in fact, other than those which can be obtained by multiplying an integer by itself) cannot be exactly expressed either as integers or as the ratios between pairs of integers, *i.e.*, either as whole numbers or as vulgar or decimal fractions (terminating or recurring), the decimals obtained in the attempt to extract such roots by the usual arithmetical process never terminating and never recurring.

This may be proved as follows:

A whole number can be obtained by multiplying together two vulgar fractions [1] only in the case in which the denominator of one fraction is a factor of the numerator of the other, and the denominator of the other a factor of the numerator of the first (the fractions both being expressed in their lowest terms); that is to say, $\frac{a}{b} \times \frac{x}{y}$ equals a whole

[1] It should be borne in mind that a decimal fraction (so long as it terminates or recurs) can always be converted into a vulgar fraction; so that the above proof holds good with reference to such decimal fractions.

number only if $b$ is a factor of $x$ and $y$ a factor of $a$; as in the case, for example, of $\frac{16}{5} \times \frac{15}{8}$, which equals 6 exactly, since $16 \div 8 = 2$, $15 \div 5 = 3$, and $2 \times 3 = 6$. Now, in the case of a fraction multiplied by itself, if this holds good it is clear that the denominator of the fraction must be a factor of its own numerator; thus, referring to the algebraical illustration above, since, in this case, $a = x$, $b = y$ and $b$ is a factor of $x$, it is clear that $b$ is a factor of $a$; the fraction, therefore, reduces to a whole number. Consequently, the product obtained by multiplying a fraction by itself cannot be a whole number. Therefore, the root of a whole number cannot be a fraction. But clearly $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, &c., are not whole numbers, whence it follows that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, &c., cannot be expressed exactly either as whole numbers or as fractions. Such roots as these are termed *surds*, or *irrational quantities*.

It is possible, however, by the usual arithmetical processes for extracting roots, to obtain the *approximate* value of the root of any number expressed as a decimal (which can, of course, be converted into a vulgar fraction if desired) to any required degree of accuracy. Thus, for example, it is found that, correct to six decimal places (*i.e.*, one-millionth part of a unit) $\sqrt{2} = 1.414214$; we know, therefore, that $\sqrt{2}$ is greater than 1.41421, but less than 1.41422. For most purposes the approximation 1.4142 is sufficiently accurate. Similarly in the case of other roots approximate values can be obtained.

**The Theorem of Pythagoras.**

**§ 16.** The square root of any number can always be represented, by a simple geometrical construction, as a straight line, the length of which, measured in terms of some other straight line, shall be equivalent to the surd in question, as we shall show later. It should be noted, however, that the like cannot be done in the case of irrational cube roots.

The discovery of the mathematical principle or law employed in the graphical representation of irrational square roots is attributed to Pythagoras; it is, therefore, known as the Theorem of Pythagoras. This theorem states that *in any right-angled triangle the square on the hypotenuse* (*i.e.*, the side opposite the right angle) *is equal in area to the sum of the squares on the two other sides*. Thus, in Fig. 1, for example, we find by actual measurement that:

AB (hypotenuse) = 3.94 in., therefore, sq. on AB = 15.52 sq. in.[2]
AC = 2.24 in., ,, sq. on AC = 5.02 sq. in.
BC = 3.24 in., ,, sq. on BC = 10.50 sq. in.
Therefore, sum of sqq. on AC and BC = 15.52 sq. in.
= sq. on AB.

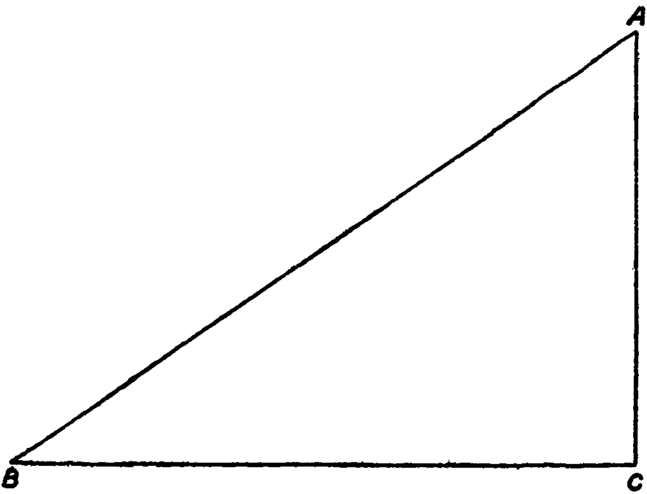


FIG. I

As a further illustration of the truth of this

[2] These values are given correct to two decimal places only, since the original measurements were not carried beyond a greater degree of accuracy. By carrying out a large number of like measurements on various right-angled triangles, we could experimentally or inductively prove the *approximate* truth of Pythagoras's theorem, which is all that is necessary for practical purposes.

theorem, we show below how the two squares on the sides of any right-angled triangle at right angles to one another, can be divided up into five pieces, which can be arranged so as to cover exactly the square on the hypotenuse.

Draw any right-angled triangle, ABC, the right angle being at C. Draw a square on each of its sides. Produce the sides of the square on the hypotenuse so as to cut the sides of the other squares in D and E respectively, the point E being in the side of the larger square (*see* Fig. 2). At E draw a straight line perpendicular to BE, cutting the adjacent side of the square in F. Cut out the two smaller squares, and divide them into five pieces along the lines AD, BE, EF. It will be found that if these five pieces are correctly arranged (as indicated by the dotted lines in Fig. 2) they will exactly cover the square on the hypotenuse, showing that the area of this square is equal to the sum of their areas, and, therefore, equal to the sum of the areas of the two squares which they together make up.[3]

The proof of the Theorem of Pythagoras

[3] There are several other methods of dividing up the squares whereby the truth of the theorem may be verified.

was probably the chief aim of the first book of Euclid's *Elements of Geometry.* The the-

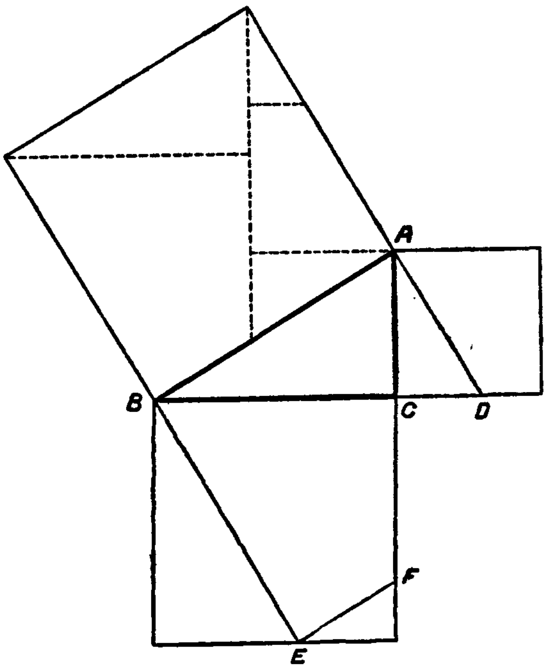


FIG. 2

orem constitutes proposition 47 of this book, to which we refer readers who are not acquainted with Euclid's proof.[4]

[4] Euclid aimed at proving the absolute, not approximate, truth of the laws of geometry, and, therefore, adopted the deductive method. It is possible, however,

**Graphical Determination of Square Roots.**

§ **17.** Now, let us draw a right-angled triangle, making the two sides which are at right angles to one another exactly 1 in. long (*see* Fig. 3). Then, clearly, the area of the square on the hypotenuse equals 1 sq. in. + 1 sq. in., *i.e.*, 2 sq. in.

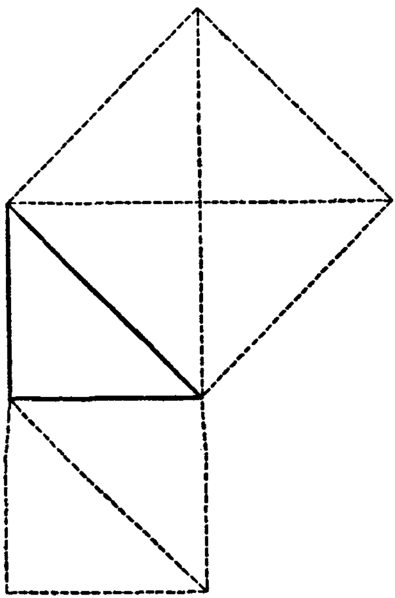

FIG. 3

(That this is actually so can readily be seen by dividing the square on the hypotenuse

to question the absolute truth of the fundamental assumptions of the Euclidean geometry ; though their practical or experimental truth is universally admitted.

into four equal triangles, each of which is one-half of a one-inch square, as shown by the dotted lines in the diagram). It follows, therefore, that the length of the hypotenuse of this triangle is $\sqrt{2}$ inches. Similarly we can represent any other irrational square

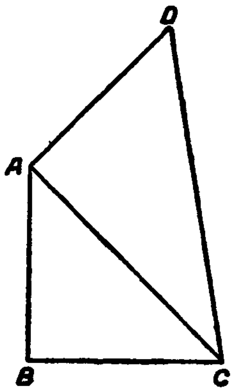


FIG. 4

root. Thus, to draw a line $\sqrt{3}$ in. in length.—Draw a right-angled triangle, making the two sides which are at right angles to one another 1 in. and $\sqrt{2}$ in. long respectively. (To draw the side which is to be $\sqrt{2}$ in. long, proceed as above, constructing first a right-angled triangle in which the sides at right angles to one another are each 1 in. long. The hypotenuse of this triangle is then used as one side of the new triangle, as in Fig. 4.) Then, clearly the area of the square on the hypotenuse of this triangle equals 1 sq. in. + 2 sq. in., *i.e.*, 3 sq. in., whence it follows that the length of its hypotenuse is $\sqrt{3}$ in. Thus, in Fig. 4, AB = BC = 1 in. Therefore, AC = $\sqrt{2}$ in. Also, AD = 1 in., whence DC = $\sqrt{3}$ in. It will be seen from this example that having drawn a line $\sqrt{x}$ units in length, we can always draw a line $\sqrt{x+1}$ units in length, by making the first line one of the sides of a right-angled triangle of which the side at right angles to it is 1 unit in length; it follows, therefore, that any and every square root can be represented by this method.

The Theorem of Pythagoras, therefore, supplies us with a graphical method for determining the approximate values of irrational square roots. All that is necessary is to draw a right-angled triangle such that the area of the square on its hypotenuse measured in terms of some convenient unit (*e.g.*, a square inch, or a square centimetre) shall be the number whose square root it is desired to extract; then, the length of the hypotenuse measured in terms of the corresponding linear unit will give the square root required. Results correct to two places of decimals can easily be obtained by this method.

**On Incommensurability.**

§ **18.** We must bear in mind, however, that in actual practice there is, of course, a limit to the accuracy with which a line can be drawn, and there is a limit to the accuracy with which it can be measured. Since the ratio between the lengths of the hypotenuse of a right-angled triangle and one of the other sides is found, in the majority of cases, to be a surd,[5] it follows that, if we were able to draw such a right-angled triangle with absolute accuracy, then, the more accurately we measured the length of its hypotenuse, our unit being the length of one of the other sides of the triangle or some length commensurable therewith, the more decimal places would make their appearance, and this would continue for ever,

[5] There are certain exceptional right-angled triangles in which the hypotenuse is commensurable with the other sides, as, for example, the right-angled triangle in which the two sides at right angles to one another are 3 and 4 units in length respectively. For, since

$$\sqrt{3^2+4^2}=\sqrt{9+16}=\sqrt{25}=5,$$

the hypotenuse of this triangle is exactly 5 units in length. The fact, that if the sides of a triangle are in the ratio 3 : 4 : 5, then the greatest angle of the triangle is a right angle, was known to the ancient Egyptians and employed by them in the erection of their temples.

the decimal never terminating and never recurring; that is to say, the hypotenuse could never be measured in terms of this unit with absolute accuracy. We see, therefore, that, as a rule, the hypotenuse of a right-angled triangle is *incommensurable* with the other sides of the triangle. It should be noted, however, that a length is never in itself incommensurable, though it may be incommensurable with some other length. $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, &c., which represent the ratio between two quantities which are incommensurable with one another, are called *incommensurables*. Consequently, we can define a surd or irrational quantity as a root which involves an incommensurable.

**Evaluation of the Base of Napierian Logarithms.**

**§ 19.** Irrational quantities are not the only incommensurables that occur in Mathematics. A very important incommensurable is the *base of natural logarithms*,[6] which is always denoted by the letter $e$. $e$ can be represented as the sum of an infinite series, that is to say, a series which continues for ever, namely:

$$1 + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \cdots$$

[6] Called also Napierian logarithms, from Napier, their discoverer. A discussion of the nature and uses of

the series being continued for ever, each term being derived from the preceding one in a similar manner. The sum of this series is equivalent to $e$, the sum of an infinite series being defined as the limiting value which the sum of its terms continually approaches as more terms are taken into account, but which it never exceeds. It can easily be shown that the sum of this series lies between 2 and 3, and the approximate value of $e$ can be calculated to any required degree of accuracy. To six places of decimals, its value is found to be 2.718282, an approximation which is easily arrived at by adding up the first eleven terms of the above series, the sum of the twelfth and following terms being less than .000001. Thus:

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logarithms is beyond the confines of the present work. It must suffice here to say that the logarithm of any number to a given base is the power to which that base must be raised to give the number in question. That is to say, if $a^n = x$, $n$ is said to be the logarithm of $x$ to the base $a$. This is written $\log_a x = n$. Thus $\log_3 9, = 2$, because $3^2 = 9$. In practical work logarithms calculated to base 10 are always employed. Logarithms to base $e$ may be calculated by means of certain infinite series; these may then be converted to base 10. In this manner tables of logarithms to base 10 are constructed, and by their aid many mathematical calculations are very much simplified. Readers unacquainted with the subject and desirous of further information may consult a text-book on practical mathematics.

$$1 = 1$$
$$\frac{1}{1} = 1$$
$$\frac{1}{1.2} = 0.5$$
$$\frac{1}{1.2.3} = 0.1666667$$
$$\frac{1}{1.2.3.4} = 0.0416667$$
$$\frac{1}{1.2.3.4.5} = 0.0083333$$
$$\frac{1}{1.2.3.4.5.6} = 0.0013889$$
$$\frac{1}{1.2.3.4.5.6.7} = 0.0001984$$
$$\frac{1}{1.2.3.4.5.6.7.8} = 0.0000248$$
$$\frac{1}{1.2.3.4.5.6.7.8.9} = 0.0000028$$
$$\frac{1}{1.2.3.4.5.6.7.8.9.10} = 0.0000003$$

Therefore $e = 2.718282$ (correct to 6 decimal places).

The incommensurable denoted by $e$ has the remarkable property that any power of it can be represented by an infinite series similar to that for $e$, but in which the numerators of the fractions of which the series consist are successive powers of the index of this power. Thus, for example:

$$e^3 = 1 + \frac{3}{1} + \frac{3^2}{1 \times 2} + \frac{3^3}{1 \times 2 \times 3} + \frac{3^4}{1 \times 2 \times 3 \times 4} + \ldots$$

and, in general:

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \times 2} + \frac{x^3}{1 \times 2 \times 3} + \frac{x^4}{1 \times 2 \times 3 \times 4} + \ldots .$$

**The Quadrature of the Circle.**

§ **20.** Another most important incommensurable is *the ratio between the diameter and circumference of a circle, i.e.*, the quantity by which the length of the diameter of a circle must be multiplied to obtain a length equal to that of the circumference. It can be shown that this ratio, which is invariably denoted by the Greek letter $\pi$ (*pi*),[7] is of constant value, *i.e.*, no matter what the size of a circle may be, the length of its diameter divided into the length of its circumference, both being measured in terms of the same unit, always gives the same quotient, namely, the quantity denoted by the letter $\pi$. It can also be shown that $\pi$ times the square on the radius of any circle is equivalent to the area of the circle. This latter fact was discovered by Archimedes, and may be proved as follows:

Suppose the circumference of the circle in Fig. 5 to be divided up into a large number ($n$) of equal parts, of which AB is one such part. Let straight lines, OA and OB, be drawn from O, the centre of the circle, to the points A and B respectively. Then, if AB be very small, that is to say, if $n$ be

[7] The length of its circumference is the *p*erimeter of a circle, and $\pi$ is the Greek letter corresponding to *p*.

very great, the figure OAB may be regarded as a plane triangle, of height equal to the radius of the circle ($r$), and base equal in length to the line AB. Now, the area of a triangle equals half the product of the lengths

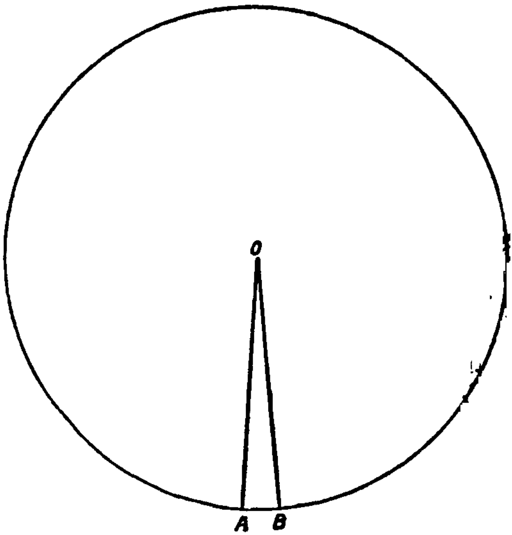

FIG. 5

of its base and height, so that the area of the figure OAB equals half the product of $r$ and the length of the line AB. Further, the length of the whole circumference equals $\pi$ times the diameter, *i.e.*, $2\pi r$; therefore, the length of the line AB equals $2\pi r \div n$; so that the area of the figure OAB equals

$\pi r^2 \div n$. But $n$ such figures make up the whole of the circle. Therefore, the area of the circle equals $\pi r^2$. It must be understood, of course, that all lengths above are measured in terms of the same linear unit, in which case all areas will be obtained in terms of the corresponding superficial unit. Since $n$ cancels out in the above proof, $n$ may be supposed to be infinitely great, in which case the distinction between the figure OAB and the corresponding plane triangle vanishes; so that the above formula for the area of a circle holds, not approximately, but exactly. It follows from the above that the determination of the correct value of $\pi$ is necessary in order to perform the quadrature or "squaring" of the circle, *i.e.*, to draw a square equal in area to a given circle; or, to express the area of a circle in terms of a given square as the unit—one of the great geometrical problems of antiquity.

**The Evaluation of "$\pi$."**

§ **21.** Archimedes was one of the first to attempt the solution of this problem in a scientific manner. By considering the circumference of a circle as intermediate in length between the perimeters of two polygons containing the same number of sides, one inscribed within the circle, the other circumscribed about it, he concluded that the value of this ratio lay between $3\frac{10}{71}$ and $3\frac{1}{7}$, *i.e.*, between 3.1408 and 3.1429. The ancient Egyptians appear, at one time, to have used the value $(\frac{4}{3})^4 = 3.16$, and the Hebrews, apparently, were content with the value 3.[8] With the development of Higher Mensuration or Trigonometry during the past three centuries, better methods for the evaluation of $\pi$ have been devised, involving the use of various infinite series. The simplest of these is due to Gregory; he showed that:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ . \ . \ . \ (\textit{ad infinitum})$$

More convenient than this, however, are the series due to Euler and Machin. It can be shown from a consideration of these series that the value of $\pi$ is incommensurable, and, therefore, the *exact* quadrature of the circle remains for ever impossible.[9] By

[8] See *The First Book of Kings*, ch. vii. v. 23.

[9] Correctly speaking, this proves only the impossibility of exactly "squaring the circle" by an arithmetical method, and it might still be thought that there may be a method of geometrical quadrature. For reasons which we cannot give here, however, mathematicians regard this also as impossible; but there are several methods of approximately-accurate geometrical quadrature. See Prof. FELIX KLEIN: *Lectures on Mathematics* (Ziwet, 1894), pp. 51–57.

means of such series, however, the approximate value of $\pi$ can be calculated to any required degree of accuracy, so the circle can be "squared" as accurately as may be desired. In 1873 Mr. Shanks carried the computation of the approximate value of $\pi$ to over seven-hundred decimal places. For most purposes the value 3.14159, which is correct to five decimal places, is sufficiently accurate, and, indeed, there are no purposes for which as many as ten decimal places are required.

The problem of "the squaring of the circle" has also, at times, attracted the attention of many persons with little knowledge of mathematics, but with considerable confidence in their powers to obtain the *exact* value of $\pi$, with the production of alleged solutions of the utmost absurdity. The value of $\pi$ correct to at least 2 decimal places can easily be obtained by direct measurement. Thus, the diameter of a cylinder can be measured by means of calipers, in the ordinary way; and the length of its circumference can be determined by wrapping wire, string, or a strip of paper around it: the ratio between these two measurements will give the value of $\pi$. Thus, in an actual case, it was found that the diameter of a certain cylinder measured 7 in., whilst its circumference was 22 in. From these measurements, since $\pi$ times the length of the diameter equals that of the circumference, the value of $\pi$ obtained is $22 \div 7 = 3\frac{1}{7} = 3.1429$, which is correct to the second place of decimals.[10] In spite of this fact, however, some of the "circle-squarers" (as distinguished from the mathematicians who have scientifically investigated the subject) have suggested values which have erred in the second or even the first decimal place.

10 The accuracy with which the value of $\pi$ is determined by such a method depends, of course, partly on the accuracy with which the cylinder is made and partly on the accuracy of the measurements.

# CHAPTER III

## ON NATURE REGARDED AS THE EMBODIMENT OF NUMBER

**The Pythagorean Theory of Numbers.**

§ 22. The theory that the Cosmos has its origin and explanation in Number is one that is inseparably connected with the name of Pythagoras, though at the present time it is extremely difficult to decide to what extent the theories ascribed to him were his own, and not due to his disciples; for Pythagoras, confining his teaching exclusively to the oral method, has left to posterity no account of his theory of the Cosmos, what writings are attributed to him being extremely meagre and most probably the work of his disciples.

The theory in question is one for which it is not difficult to account if we take into consideration the nature of the times in which it was formulated. The Greek of the period, looking upon Nature, beheld no picture of harmony, uniformity and fundamental unity. The outer world appeared to him rather as a discordant chaos, the mere sport and plaything of the gods. The theory of the uniformity of Nature—that Nature is ever like to herself—the very essence of the modern scientific spirit, had yet to be born of years of unwearied labour and unceasing delving into Nature's innermost secrets. Only in Mathematics—in the properties of geometrical figures, and of numbers—was the reign of law, the principle of harmony, perceivable. Even at this present day when the marvellous has become commonplace, that property of right-angled triangles we have already discussed in the preceding chapter comes to the mind as a remarkable and notable fact: it must have seemed a stupendous marvel to its discoverer, to whom, it appears, the regular alternation of the odd and even numbers, a fact so obvious to us that we are inclined to attach no importance to it, seemed, itself, to be something wonderful. Here in Geometry and Arithmetic, here was order and harmony unsurpassed and unsurpassable. What wonder then that Pythagoras concluded that the solution of the mighty riddle of the Universe was contained in the mysteries of Geometry? What wonder that he read mystic meanings into the laws of Arithmetic, and believed Number to be the explanation and origin of all that is?

We say, There can be no wonder at all. The Pythagorean theory was the natural outcome of an erroneous view-point, itself resulting from the fact that philosophers as yet had not deigned to investigate the phenomena of external Nature, but applied themselves solely to the more abstract studies.

**Nature as the Embodiment of Number.**

§ **23.** Whilst criticising the Pythagorean theory of the Cosmos, however, we must not forget that there is undoubtedly a considerable element of truth in it. It is this—to whatever part of the material world we turn, we are confronted with the ubiquity of Number: the forms of crystals, beautiful in their symmetry; the combinations of the atoms, and even, it appears, the constitution of the atoms themselves—all are expressible in terms of number. In the biological sciences it is true that Mathematics does not, as yet, play so important a part as in the physical sciences, but even here its importance is being recognised. The arrangement of the leaves about the stem in various plants, to take but one fact out of many, is found to follow a mathematical law and may be expressed as a numerical ratio. In brief, the whole material world is the embodiment of number. As says Dr.

Ennemoser, " The signification of the Pythagorean numeral theory is, that numbers contain the elements of all things, and even of the sciences. It was clearly seen that everything in nature may be reduced to numeral conditions." " We cannot," he remarks later, " be blind to the fact that, in such a theory of numerals, a real and profound signification may be contained, and not alone an idle speculation or fantastical subtlety. For, through the wonderful progress of modern chemistry, the old axiom that determined numerical conditions govern the material world has gained an unexpected signification. Stoichiometry shows indisputably in the combination of atoms, a regularity of number as strictly observed by God in the minutest forms, as in the majestic nature of the heavens." [1] More recent researches in the domains of Chemistry and Physics (Dr. Ennemoser wrote in 1843) fully confirm the validity of this argument.

**§ 24.** The question of incommensurables, however, arises. If number be all-potent in Nature, how comes it that there are ratios which numbers fail to represent exactly ? In

[1] JOSEPH ENNEMOSER : *The History of Magic* (translated by William Howitt, 1854), vol. i. pp. 394, 399 and 400.

order to reply to this question we must look a little more closely into the signification of incommensurability.

**The Significance of Incommensurability.**

Briefly, the reason why numbers fail to represent exactly the ratios between certain lengths, such as the diameter and circumference of a circle, is that, whereas length is continuous, number is essentially *discontinuous*. As Sir Oliver Lodge puts it, "Numerical expression is more like a staircase than a slope: it necessarily proceeds by steps: it is discontinuous."[2] This is apparent with regard to the integers or whole numbers, and a little consideration will convince us that the whole numbers are not really connected by the fractions intermediate in value between them and rendered continuous thereby; for between any two fractions, however slightly they may differ, another fraction, differing from both of these two less than they differ from one another, can always be inserted; and between each pair of these fractions others can be inserted likewise, and so on *ad infinitum*. It is clear,

[2] Sir OLIVER LODGE: *Easy Mathematics* (1910), p. 187. (We are much indebted to Sir Oliver Lodge's lucid treatment of the subject at present under discussion.)

therefore, that between any two fractions whatever, an infinite number of other fractions can always be inserted, so that by the insertion of fractions the integers can never be rendered continuous. Sir Oliver Lodge gives an excellent illustration of this fact. He says: "Look at the divisions on a foot rule; they represent lengths expressed numerically in terms of an arbitrary length taken as a unit: they represent, that is to say, fractions of an inch; they are the terminals of lengths which are numerically expressed; and between them lie the unmarked terminals of lengths which cannot be so expressed. But surely the subdivision can be carried further. . . . Why not divide originally into tenths and then into hundredths, and those into thousandths, and so on? Why not, indeed? Let it be done. It may be thought that if we go on dividing like this we shall use up all the interspaces and have nothing left but numerically expressible magnitudes. Not so, that is just a mistake; the interspaces will always be infinitely greater than the divisions. For the interspaces have all the time had evident breadth, indeed they together make up the whole rule; the divisions do not make it up, do not make any of it, however numerous they are. For how wide are the divisions? Those we make, look, when examined under the microscope, like broad black grooves. But we do not wish to make them look thus. We should be better pleased with our handiwork if they looked like very fine lines of unmagnifiable breadth. They ought to be really *lines*—length without breadth; the breadth is an accident, a clumsiness, an unavoidable mechanical defect. They are intended to be mere divisions, subdividing the length but not consuming any of it. All the length lies between them; no matter how close they are they have consumed none of it; the interspaces are infinitely more extensive than the barriers which partition them off from one another; they are like a row of compartments with infinitely thin walls.

"Now all the incommensurables lie in the interspaces; the compartments are full of them, and they are thus infinitely more numerous than the numerically expressible magnitudes."[3] It follows, therefore, as Sir Oliver Lodge points out, that the chances are infinity to one against any two given lengths being commensurable with one another.

§ **25.** Whilst, however, this is true of mathematical lengths—lengths in the ab-

[3] *Loc. cit.*, pp. 189–191.

stract—for mathematical length is continuous, the question arises whether it is likewise true of lengths as embodied in material existence. From a distance a powder appears to be continuous, closer examination shows that it consists of separated particles. And it appears that this is true of all material objects: all material things have a grained structure; the most compact and dense forms of matter consist of separated molecules, and these molecules of separated atoms. It follows, therefore, that no material length (provided that all forms of matter are ultimately reducible to one and the same unit [4]) can really be incommensurable with any other material length, with the possible exceptions noted below. In the right-angled triangle in Fig. 3, for example, however carefully

**The Discontinuity of Matter.**

[4] Otherwise it might happen that one sort of atom might be incommensurable with some other sort. Even admitting the above it might still be argued that, as the size of any material body depends not so much upon the actual size of its constituent particles as upon the amplitude of their motion, the dimensions of a body consisting of one sort of material substance may be incommensurable with the dimensions of a body consisting of some other sort of material substance or of the same material substance at a different temperature. The main contention above, however, will, we think, be admitted.

it may have been drawn, the hypotenuse simply cannot be exactly $\sqrt{2}$ times one of the other sides; for all the sides of this triangle are material lengths; they are thin, narrow prisms of carbon, and each one of them must consist of a definite whole number of carbon atoms, for not less than one atom of any material element can exist. It follows, therefore, that the ratio between the hypotenuse and either side of this triangle if worked out as a decimal, did we know the number of carbon atoms in each line, would ultimately terminate or recur, although it might be an alarmingly long decimal; for the reason that an integer, no matter how great it be, is always commensurable with every other integer, and as we have seen above, the lengths of all the sides of the triangle in question, if measured in terms of the length[5] of the carbon atom as unit, would be expressible as integers. And this argument holds good of all other material lengths and other quantities, for the most perfect material line conceivable consists of a row of *discontinuous* atoms. Moreover, if the electronic theory of matter be true, it must be impossible for actual lengths incom-

[5] *I.e.*, the apparent length, the amplitude of the motion of the atom being taken into account.

mensurable one with the other to exist in a physical sense on the surface of, or within, an atom itself; for according to this theory the atoms are composed of separated electrons (or units of negative electricity), just as matter is composed of separated atoms.

**The Continuity of the Ether.**

§ **26.** The discontinuity of matter necessitates the assumption of a perfectly continuous medium, an absolute plenum, filling all space; for if there were not such a medium, all action would be action at a distance, action across empty space, which is unthinkable. This medium modern science calls the "Ether of Space." It follows, therefore, that the theory, sometimes expressed, that the Ether is atomic in structure, reduces to a contradiction in terms. Or, if it be argued that the Ether should be defined as the medium whereby light, radiant heat, &c., are transmitted, and that this medium is atomic in structure and therefore discontinuous, then we must postulate an ether beyond the Ether to fill up the interspaces between the particles of the latter. However, the phenomena connected with the transmission of light and other etheric phenomena are in better agreement with the view that the medium whereby light is transmitted is of a continuous nature. Here, then, in the ether, incommensurability acquires a real physical significance. We can conceive of a prism of ether of which the cross-section is a right-angled triangle with a hypotenuse incommensurable with its other sides ; we cannot conceive of a similar material prism. We can conceive of a *perfect* cylinder of ether, but a material cylinder must necessarily deviate somewhat from geometrical exactitude. We can conceive of lengths incommensurable one with the other existing in a physical sense in the ether ; but when we reach the ether we have transcended the world of matter in the more restricted use of the term. So long as we confine our attention to the world of matter, so long do we find the reign of number supreme. In a certain sense, the realm of the continuous is representative of Unity, whilst the realm of the discontinuous, the world of matter, is representative of the diversity of Number. For in the realm of the continuous, one only exists—itself, and although we may consider all numbers as existing here potentially, they have not yet been born into actuality ; for although we may consider the realm of the continuous as consisting of diverse parts representative of all numbers and quantities (including incommensurables), there is nothing to distinguish or divide one part from another. Accepting the doctrine of modern science that the discontinuous originates from the continuous, the world of matter from the Ether, we may say that the birth of matter is the birth of numbers from potentiality into actuality.

**The Absolute Unit of Measurement.**

§ **27.** Before concluding this chapter there is one other thought that it may be worth while to follow out. There is something very arbitrary about number if it be regarded as a ratio merely. The number expressing some length when the unit of measurement employed is the foot is quite different from the number expressing the same length when some other unit of measurement, say, the inch, or the yard, or the metre, is used instead; and the same holds good for all other measurements. The length of a thing in feet, its weight in pounds, &c., all such are purely arbitrary numbers. And if we employ the so-called "absolute units," the case is not fundamentally altered. The *metre*, for example, was supposed to be exactly the 10,000,000th part of the distance from the earth's equator to its pole when the French Government made it the compulsory unit of length in 1801, but more recent determinations have shown it to differ slightly from this. Supposing, however, that the metre were exactly the 10,000,000th part of the distance from the earth's equator to its pole, it would still be a purely arbitrary unit on any other planet, and the same can be said of all the other so-called "absolute units." But if all the varying forms of matter arise from some one common unit, if all material bodies are systems of one and precisely the same sort of particle, then from the properties of this ultimate particle (the electron, it may be) could be derived a system of genuinely absolute units for all material measurements whatever. Thus, its diameter would constitute the unit of length, its inertia the unit of inertia, and so on. For practical purposes, of course, these units would be most inconveniently small, but that is not the point—for practical purposes the metric system of weights and measures does very well. The point is rather that all the manifold differences between the various things of the material world reduce, in the ultimate analysis, to differences in the number of ultimate particles composing them, and differences in the arrangement and motion of these particles, all of which are expressible mathematically. From this point of view, every material thing is capable of being represented by numbers which have a more than arbitrary meaning, and the doctrine that all material Nature is the embodiment of Number acquires a real significance.

# CHAPTER IV

## ON NEGATIVE QUANTITIES

**Definition of a Negative Quantity.**

§ **28.** A *negative* quantity may be defined briefly as a quantity *less than nothing*. Such quantities are indicated by means of the *minus sign* ( $-$ ), ordinary or *positive* quantities (*i.e.*, quantities greater than nothing) being indicated by the *plus sign* ( $+$ ), which is, however, generally omitted. For example, $-2$ (minus two) means a quantity as much below zero as 2, *i.e.*, $+2$ (plus two), is above zero; and, in general, $-x$ means a quantity as much below zero as $x$ or $+x$ is above zero. Suppose, for instance, we were asked to find the result of subtracting 6 from 2, *i.e.*, the value of $2-6$. It is clear that, since 2 subtracted from 2 gives zero, and since 6 is greater than 2 by 4, the result of subtracting 6 from 2 will be 4 less than zero, *i.e.*, $-4$. It may here very possibly be objected by the reader that such an operation as the above is impossible, and that a negative quantity is an absurdity. In a certain sense, as we shall see later, this is true; but in another sense it is not. Very many cases occur in which we must consider the subtraction of a quantity from one less than itself as possible, and in which it is found convenient to employ quantities less than zero. A few illustrations, however, will make the whole matter perfectly plain.

**Examples of the Uses of Negative Quantities.**

**§ 29.** (*a*) Take in the first case the idea of a *debt*. Suppose the whole of a man's belongings amount in value to £500 and the whole of his debts to £300, then, clearly, he is worth £500 − £300, *i.e.*, £200. If, however, his debts also amount to £500 he must be worth £500 − £500, *i.e.*, nothing. But suppose, now, that his debts amount to £700. Clearly it would not be correct in this case to say merely that he is worth nothing, for he is now worse off than in the second case considered above. In that case his assets exactly balance his liabilities, but now he has a balance of £200 on the wrong side: he is worth £200 less than nothing; in other words, he is worth £500 − £700, *i.e.*, £ −200.

(*b*) Secondly, take the idea of *direction*. Suppose, for example, three towns, A, B, C, to lie so that town A is situated 4 miles due south of town B, and town C four miles due north of town B. Then considering the position of each town with reference to town B, whose position is taken as the zero, if we call that of town C $+4$ miles, since A and C lie on opposite sides of B, we must call the position of town A $-4$ miles.

(*c*) A further example of the use of negative quantities is afforded by the measurement of temperature. On the Fahrenheit thermometer (commonly used in England) the zero-point indicates a certain temperature which was the lowest attainable by Fahrenheit; whilst on the Centigrade thermometer (invented by Celsius, and employed on the Continent and for scientific work generally) the zero-point indicates the temperature at which ice melts. Temperatures far below either of these zero-points, however, are attainable, and to indicate these negative quantities are employed. Thus a temperature of $-100°$ C.[1] means a temperature as

[1] $1°$ C. represents a temperature of 1 degree on the Centigrade scale and, similarly, $1°$ F. represents a temperature of 1 degree on the Fahrenheit scale. On both scales the temperatures at which ice melts and at which water boils when the pressure of the atmosphere is equivalent to a column of mercury 760 mm. high are taken as the fixed points. On the Centigrade scale these two points are numbered $0°$ and $100°$ respectively, whilst on the Fahrenheit scale they read

much below the Centigrade zero as 100° C. (the boiling-point of water) is above it.

**Addition of Negative Quantities.**

**§ 30.** We must now turn our attention to the consideration of the addition of negative quantities. It is clear that if a man contracts a debt receiving nothing in return (*e.g.*, a gambling debt) he is the worse off by the amount of this debt. We see, therefore, that the addition of a negative quantity is equivalent to the subtraction of the corresponding positive quantity. Thus $-4$ added to 6, *i.e.*, $6+(-4)$, gives 2, as also does $+4$ subtracted from 6, *i.e.*, $6-(+4)$. Therefore, $6+(-4)$, $6-(+4)$ and $6-4$ are equivalent expressions; and, in general: [2]

$$x+(-y)\equiv x-(+y)\equiv x-y$$

32° and 212°. It follows therefore that a rise or fall in temperature of 5° on the Centigrade scale is equivalent to a rise or fall of 9° on the Fahrenheit scale. Allowing for the difference in the zero-points, the formulæ for converting measurements on one scale into measurements on the other are, therefore,

$$C=\frac{5(F-32^\circ)}{9}, \qquad F=\frac{9C}{5}+32^\circ$$

where C and F are corresponding temperatures on the Centigrade and Fahrenheit scales respectively.

[2] The symbol "$\equiv$" is here used, as frequently, in the sense of "is identical with," that is to say, "is equal to, no matter what values be given to the algebraical symbols employed."

Further, since it is clear that all a man's debts go to decrease the balance on the right side of his accounts or to increase the balance on the wrong side, whilst the effect of all his gains is precisely the opposite of this, we see that to find the sum, or total value, of a number of quantities, some of which are positive and some negative, we must first of all find the numerical sum of all the positive quantities and the numerical sum of all the negative quantities separately, and then calculate the difference between them prefixing the sign of the greater.

Thus :

$$(a)\quad 18+6-15-3+2-16=26-34$$
$$=-8$$

whilst

$$(b)\quad -18-6+15+3-2+16=34-26$$
$$=8\ (i.e.,\ +8)$$

In the case of a number of negative quantities, their sum will, of course, be their numerical sum with the negative sign prefixed. Thus, for example, $-6-5-10=-21$, &c.

§ **31**. The illustration of a debt will also help to make plain the result of the subtraction of negative quantities. For if we remove or take away a man's debts, this is precisely the same as if we gave the man the amount of his debts. We see, therefore, that the subtraction of a negative quantity is equivalent to the addition of the corresponding positive quantity. Thus, $-4$ subtracted from 6, *i.e.*, $6-(-4)$, gives 10, as also does $+4$ added to 6, *i.e.*, $6+(+4)$, or $6+4$. Therefore, $6-(-4)$, $6+(+4)$ and $6+4$ are equivalent expressions, and, in general:

**Subtraction of Negative Quantities.**

$$x-(-y) \equiv x+(+y) \equiv x+y$$

In the case of the subtraction of a negative quantity from a negative quantity the same rule holds good, but as we now have to add a positive quantity to a negative one the result will be the difference between the two quantities with the sign of the greater prefixed. Thus, for example,

$$-6-(-4) = -6+4 = -2$$

and

$$-6-(-8) = -6+8 = +2.$$

§ **32.** We must now turn our attention to a consideration of the multiplication of negative quantities. Let us consider the several results of multiplying the sum of two quantities, (i.) by itself, and (ii.) by the difference between the two quantities, and of multiplying the difference

**Geometrical Proofs of Multiplication Formulæ.**

between two quantities, (iii.) by itself, and (iv.) by the sum of the two quantities. Representing the two quantities by $a$ and $b$, respectively, the problem before us is to determine the value of the following products :[3]

$$\begin{array}{rl} \text{(i.)} & (a+b)\,(a+b) \\ \text{(ii.)} & (a+b)\,(a-b) \\ \text{(iii.)} & (a-b)\,(a-b) \\ \text{(iv.)} & (a-b)\,(a+b) \end{array}$$

of which (ii.) and (iv.) are clearly identical. We shall approach this problem from the geometrical point of view. Consider Figs. 6–8 on pp. 85–87. For the strict geometrical construction of these figures and proofs of the propositions stated below, reference must be made to Euclid's *Elements of Geometry*, Book II., props. 4, 5 and 7, or any other text-book of Geometry covering the same ground. The following facts, however, can be verified by actual measurement, and thereby the truth of each of the following propositions may be experimentally or inductively proved. In Fig. 6 call the length of the

[3] The multiplication sign is usually omitted in algebraical expressions, mere juxtaposition indicating that the quantities so placed are to be multiplied together. A bracket indicates that whatever is enclosed within is to be treated as a whole.

line AC $a$, and that of the line CB $b$. Then, the area of the square AE $=(a+b)^2$; the area of the rectangle AG $=ab$; the area of the square CK $=b^2$; the area of the square

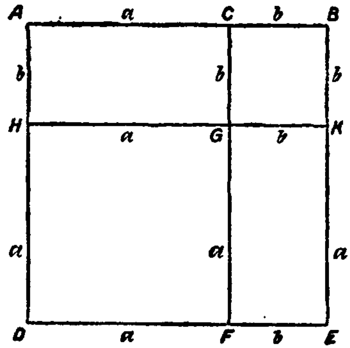


FIG. 6

HF $=a^2$; and the area of the rectangle GE $=ab$. But the area of the square AE equals the sum of the areas of the figures AG, CK, HF and GE. Therefore,

$$(a+b)^2=ab+b^2+a^2+ab;$$

that is to say,

$$(a+b)(a+b)=a^2+2ab+b^2.$$

In Fig. 7 call the length of the line AP (which equals the length of the line PB) $a$, and that of the line PQ $b$. Then, the area of the rectangle AF $=(a+b)(a-b)$; but the rectangle AH equals the rectangle QD, therefore the area of the figure PFD = $(a+b)(a-b)$. Also, the area of the square PD $=a^2$, and the area of the square HE $=b^2$. But the area of the figure PFD equals the

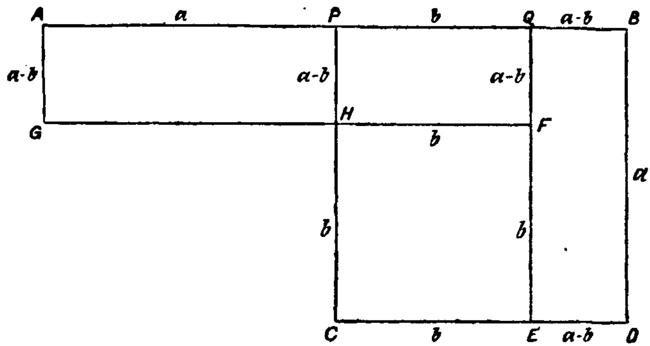


FIG. 7

area of the square PD less the area of the square HE. Therefore:

$$(a+b)(a-b)=a^2-b^2$$

In Fig. 8 call the length of the line AB $a$, and that of the line CB $b$, so that the length of the line AC is $a-b$. Then, the area of the square AE $=a^2$, the area of the square CK $=b^2$, the area of the square HF $=(a-b)^2$ and the area of the rectangle AK = the area of the rectangle CE $=ab$. Now, the area of the square AE plus the area of the square CK equals the area of the rectangle AK plus the area of the rectangle CE plus the area of the square HF. Therefore, the area of the square HF equals the area of the square AE plus the area of the square CK less twice the

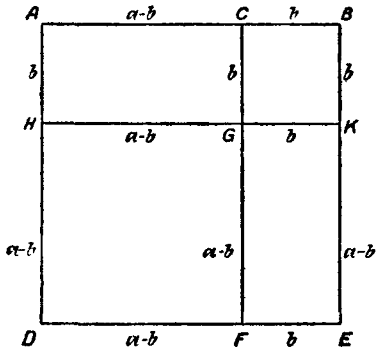


FIG. 8

area of the rectangle AK. Therefore,

$$(a-b)^2 = a^2 + b^2 - 2ab.$$

That is to say,

$$(a-b)(a-b) = a^2 - 2ab + b^2$$

**Multiplication of Negative Quantities.** § 33. In actual practice these several results are obtained as follows. In the case of the first:

$$\begin{array}{ll} a + b & \\ \underline{a + b} & \\ a^2 + ab & \text{(product of } a \text{ and } a+b\text{)} \\ \underline{\quad + ab + b^2} & \text{(product of } b \text{ and } a+b\text{)} \\ a^2 + 2ab + b^2 & \text{(sum of the two products).} \end{array}$$

the process being similar to "long multiplication" in Arithmetic. Now, in the second case, namely, $(a+b)(a-b)$, the product of $a$ and $a+b$ is $a^2+ab$, as above, and we have shown geometrically that the sum of the two products must come to $a^2-b^2$, clearly, therefore, the second product (*i.e.*, the product of $-b$ and $a+b$) will be the difference between $a^2-b^2$ and $a^2+ab$, *i.e.*, $-ab-b^2$, the complete sum being:

$$\begin{array}{ll} a+b & \\ a-b & \\ \hline a^2+ab & \text{(product of } a \text{ and } a+b) \\ \quad -ab-b^2 & \text{(product of } -b \text{ and } a+b) \\ \hline a^2 \qquad -b^2 & \text{(sum of the two products)} \end{array}$$

We see, therefore, that *the product of a positive quantity and a negative quantity is itself a negative quantity*, equal in magnitude (but opposite in sign) to the product of the two corresponding positive quantities; thus

$$(+a)\times(-b) = -ab, \text{and} (+b)\times(-b) = -b^2, \text{\&c.}$$

Coming now to the third case, namely, $(a-b)(a-b)$, we see that the product of $a$ and $a-b$ must be $a^2-ab$, and we have shown geometrically that the sum of the two products must come to $a^2-2ab+b^2$, clearly, therefore, the second product (*i.e.*, the product of $-b$ and $a-b$) will be the difference between $a^2-2ab+b^2$ and $a^2-ab$, *i.e.*, $-ab+b^2$, the complete sum being:

$$\begin{array}{ll} a\ -b & \\ \underline{a\ -b} & \\ a^2-ab & \text{(product of } a \text{ and } a-b) \\ \underline{\quad -ab+b^2} & \text{(product of } -b \text{ and } a-b) \\ a^2-2ab+b^2 & \text{(sum of the two products)} \end{array}$$

We see, therefore, that *the product of two negative quantities is itself a positive quantity*, equal in magnitude to the product of the two corresponding positive quantities, thus

$$(-b)\times(-b)=+b^2.$$

Similarly,

$$(-a)\times(-b)=+ab.$$

Precisely the same conclusions can be arrived at by a different line of reasoning, as follows. Consider the following identities which have been obtained above:

$$(a+b)(a+b)\equiv a^2+2ab+b^2 \quad . \quad . \text{ (i.)}$$
$$(a+b)(a-b)\equiv a^2-b^2 \quad . \quad . \quad . \text{ (ii.)}$$
$$(a-b)(a-b)\equiv a^2-2ab+b^2 \quad . \quad . \text{ (iii.)}$$

We can, it is true, obtain geometrical representations of identities (ii.) and (iii.) only in those cases in which $a$ is greater than $b$, but by the principle of the continuity of mathematical law we must regard these identities as holding good for all conceivable values of $a$ and $b$. Suppose $a=0$, then the following results follow:

$$(+b)\times(+b) = +b^2$$
$$(+b)\times(-b) = -b^2$$
$$(-b)\times(-b) = +b^2$$

in agreement with the above conclusions. We may summarise these results by stating that *the product of two quantities with like signs (whether plus or minus) is always positive, whilst the product of two quantities with unlike signs is always negative.* It should be noticed that every positive quantity, therefore, has *two* square roots, equal in magnitude but opposite in sign. Thus, the square roots of $+9$ are $+3$ and $-3$, since $(+3)\times(+3)=+9$ and $(-3)\times(-3)=+9$. The double sign $\pm$ is usually employed to show this, thus, we write $\sqrt{9} = \pm 3$, and in general, $\sqrt{x^2} = \pm x$.

**Division of Negative Quantities.**

§ **34**. From the above results we can at once obtain the various formulæ in the case of division. Thus:

$(+a)\times(+b) = +ab$; therefore
$$+ab \div +a = +b$$
$(+a)\times(-b) = -ab$; therefore
$$-ab \div +a = -b$$

$$(-a)\times(+b) = -ab \text{ ; therefore}$$
$$-ab \div -a = +b$$
$$(-a)\times(-b) = +ab \text{ ; therefore}$$
$$+ab \div -a = -b$$

We see, therefore, that, similarly as in the case of products, the quotient of two quantities with like signs (whether plus or minus) is always positive, whilst the quotient of two quantities with unlike signs is always negative.

**Negative Quantities: not Embodied in Nature.**

§ **35.** We must now inquire in what sense, if any, we can regard negative quantities as having a real existence. Now, if we examine the examples of the uses of negative quantities given above (§ 29), we shall come to the conclusion that negative quantities have no existence as embodied in the physical realm. To take the first illustration there given; the idea of a debt involves that of moral obligation, and, therefore, transcends the realm of the physical. In a strictly physical sense, although it is quite possible to conceive of a man being worth nothing, it is altogether impossible to conceive of a man being worth less than nothing. Morally speaking, a man may be worth anything less than nothing, that is to say, he may be in debt for any amount with nothing wherewith to pay his debt, but with the full determination or under the obligation to do so should circumstances render this possible ; but, physically speaking, when he is worth nothing, he can be worth nothing less. Coming to the second and third illustrations referred to above, we see that in each case our zero is a purely arbitrary one. Town B is not really the beginning of space ; neither 0° F. nor 0° C.[4] is the absolute zero of temperature indicative of entire absence of heat. It must not be supposed that there is any such thing as "negative heat," though this is the way in which many unscientific people appear to regard "cold." Cold is merely a less degree of heat ; the coldest bodies, those whose temperatures are indicated by negative quantities on the ordinary thermometric scales, do possess some heat. For certain reasons, into which we cannot here enter, it is concluded that $-273°$ C.[5] represents the absolute zero of temperature, entire absence of heat, below which no lower temperature is possible.

§ **36.** In many physical measurements, negative quantities are of very great utility,

[4] See footnote 1 on p. 80.

[5] The lowest temperature yet reached is $-268\frac{1}{2}°$ C., the boiling-point of liquid helium. This temperature was obtained by Prof. Onnes, for an account of whose work see *Journal Chem. Soc.*, vol. xciv. (1908), pt. ii., p. 944.

since they allow of such measurements being made with reference to some arbitrary zero-point, and it is often very convenient — sometimes the only method possible—to work from such an arbitrary zero. In certain cases the absolute zero is doubtful or unknown, as in the case of electrical potential, in the measurement of which the electrical potential of the earth is taken as the zero; in other cases, *e.g.*, in the measurement of distance or position, the absolute zero is inconceivable. Of course, measurements which refer to an arbitrary zero cannot be regarded as *absolutely* true; the numbers expressing such measurements cannot be regarded as embodied in the things so measured. This would actually be the case only if the measurements in question referred to an absolute zero and if, as we have pointed out in §27, such measurements were expressed in terms of absolute units derived from the ultimate particle of all material existence. But measurements which refer to an arbitrary zero (which must, of course, be the same for all measurements concerned) are perfectly true relative to one another, and lead to no erroneous conclusions so long as their relative nature is not

**Arbitrary Zero-points in Physical Measurements.**

forgotten. Now, in working with an arbitrary zero, the possibility must always remain of measurements falling below this point, but no difficulty is experienced because such measurements can be accurately represented by negative quantities. If it were not for such negative quantities, however, either we should be compelled to employ absolute zeroes, which in many cases, as we have pointed out above, is impossible, or else we should be obliged to be continually shifting our zero-points to avoid measurements falling below them. The great utility and importance of negative quantities so far as physical science is concerned is, therefore, evident.[6]

**The Reality of Negative Quantities.**

§ 37. In reply to the question, Can we regard negative quantities as in any sense physically real? it appears that the most satisfactory answer is to the effect that we may regard a negative quantity as representing a physically real *operation*, at least,

[6] In Geometry, distances measured to the right of the arbitrary zero-point chosen are regarded as positive, distances measured to the left as negative. There is no inherent reason for this, we could equally well regard distances measured to the right as negative, those measured to the left as positive; but the above convention is invariably adopted, and is, therefore, very convenient.

physically real in a potential sense. To put this in other words, a negative quantity represents an operation which under certain conditions can always be actually carried out on the physical plane. Thus, for example, $-6$ represents an operation which can actually be carried out with reference to what is represented by the number 12 (say), but which cannot be actually carried out on the physical plane with reference to what is represented (say) by the number 3.

In our study of various mathematical quantities we have seen that the ordinary numbers, fractional as well as integral, may be regarded as having a real existence embodied in the material world, or, putting this in other words, we have seen that all things of the material world are expressible by means of numbers. We have seen further, however, that other quantities besides the ordinary or common numbers are involved in the study of Mathematics; and we have sought to justify the use of such quantities and to indicate wherein their reality lies. We have already thus considered incommensurable and negative quantities: in the following chapters we shall consider certain quantities which are known as "imaginary quantities," and which completely transcend the realm of the physical, and whose existence appears to be purely ideal; we shall show further for what reasons and in what manner we believe these quantities may be employed in the elucidation of certain metaphysical problems.

## CHAPTER V

### ON "IMAGINARY" QUANTITIES

**The Square Roots of Negative Quantities.**

**§ 38.** We have seen already in § 33 that the product of two quantities with like signs (whether positive or negative) is itself always *positive*, hence the squares of all positive and negative quantities, *i.e.*, of all quantities we have as yet considered, are positive. *The symbol* "$\sqrt{-1}$," which frequently occurs in mathematical investigations, *expresses, therefore, some new quantity which cannot be regarded as either positive or negative.* "When the quantity under the radical sign is negative, we can no longer consider the symbol $\sqrt{}$ as indicating a possible arithmetical operation; but just as $\sqrt{a}$ may be defined as a symbol which obeys the relation $\sqrt{a} \times \sqrt{a} = a$, so we shall define $\sqrt{-a}$ to be such that $\sqrt{-a} \times \sqrt{-a} = -a$, and we shall accept the meaning to which this assumption leads us." [1]

[1] H. S. Hall and S. R. Knight: *Higher Algebra* (fourth edition), p. 74.

For the sake of simplicity the quantity $\sqrt{-1}$ is generally written $i$. Now, it can at once be seen that, using this unit in place of 1, a new and infinite series of quantities can be obtained, namely, $i$, $2i$, $3i$, $4i$ . . . &c., exactly corresponding to our ordinary system of numbers 1, 2, 3, 4 . . . &c. It can, moreover, be shown that the square root of any negative quantity can be expressed in terms of $i$. For, by definition:

$$\begin{aligned} \sqrt{-1}\times\sqrt{-1} &= -1 \\ \textit{i.e.},\ i\times i &= -1 \\ \text{Therefore, } i\sqrt{a}\times i\sqrt{a} &= -1\times a \\ \text{that is to say, } (i\sqrt{a})^2 &= -a \end{aligned}$$

so that the product $i\sqrt{a}$ may be regarded as equivalent to the quantity $\sqrt{-a}$. Thus $\sqrt{-4}=i\sqrt{4}=2i$, &c.

**The Square Root of a Negative Quantity Physically Unintelligible.**

**§ 39.** Now, these new quantities are incapable of physical embodiment, they are meaningless on the physical plane. The value of an incommensurable quantity can be determined to any required degree of accuracy expressed either as a vulgar or decimal fraction—we know, for example, that $\pi$ is greater in value than 3.141 and less than 3.142 (*see* § 21)—but the quantity represented by the symbol $\sqrt{-1}$ or $i$ is *absolutely* inexpressible in terms of our ordinary system of numbers; we cannot say, for example, that $i$ is either less or greater than 1—such comparison is impossible, $i$ is neither less than, equal to, nor greater than 1. Further, we cannot even regard $i$ as positive or negative, for as we have seen already it cannot be either. It is true that such symbols as $-i$, $-ia$, &c., are employed, and that a like distinction is drawn between $-i$ and $+i$ as between $-1$ and $+1$. By $-i$, however, we understand the product of $i$ and $-1$, and by $-ia$ the product of $i$ and $-a$, whilst $+ia$ or $ia$ represents the product of $i$ and $+a$. It will be seen, therefore, that the distinction between positive and negative resides rather in the quantity by which $i$ is multiplied than in the quantity $i$ itself. It should further be noticed that, although we may regard $i$ as representing an operation, it is not an operation that is physically intelligible, for the result is an ordinary positive or negative quantity, and therefore, physically intelligible, only in the case in which the operation is performed on something itself not physically intelligible, namely, $i$ itself, or a product of $i$ and some ordinary quantity; thus, whilst $i$

or $ai$ multiplied or divided by $i$ yields an ordinary negative or positive quantity, an ordinary quantity multiplied or divided by $i$ yields one involving $i$.[2]

**Powers of *i*.**

**§ 40.** It will not be without interest to determine the values of the various powers of $i$. Thus:

$$\begin{array}{lllll}
i^1 & & & & = i \\
i^2 = i \times i & & & & = -1 \\
i^3 = i^2 \times i & = -1 \times i & & & = -i \\
i^4 = i^2 \times i^2 & = -1 \times -1 & & & = 1 \\
i^5 = i^4 \times i & = 1 \times i & & & = i \\
i^6 = i^4 \times i^2 & = 1 \times -1 & & & = -1 \\
i^7 = i^4 \times i^3 & = 1 \times -i & & & = -i \\
i^8 = i^4 \times i^4 & = 1 \times 1 & & & = 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

It will be seen from the above that the values $i$, $-1$, $-i$, $1$, occur over and over again in a never-ending cycle.

**The Reality of "Imaginary" Quantities.**

**§ 41.** It may be argued by the non-mathematical reader that all this is mere nonsense: that $i$ is an absurdity — a thing which has no existence. And, in fact, for the sake of convenience, ordinary numbers are always referred to as "*real*," whilst those quantities which have $i$ for their unit, *i.e.*,

[2] See "Note on the Geometrical Representation of Imaginary Quantities" at end of Chapter V.

all quantities which can be expressed as the square root of a negative quantity, are said to be "*imaginary*."[3] But it cannot be maintained that this nomenclature is altogether fortunate. The term "imaginary" in its strictly psychological sense means something which exists in imagination, *i.e.*, something which exists as a mental image. But the fact must be confessed that we can form no *mental image* of the quantity represented by $i$, though this must not be taken to mean that we can form no *idea* of this quantity altogether. In the less accurate, but more common, use of the term "imaginary," it stands for something unreal, something impossible, and this meaning is suggested in the present connection by the fact that the common numbers are denominated "real." Now, this word "real" is very elusive in meaning. If we limit its sphere of application to the realm of the physical, then let us at once admit that "imaginary" quantities being absolutely "unphysical," are therefore absolutely unreal; but the question arises whether we are justified in this limitation. The thing that exists ideally, that is to say,

[3] The sum of, or difference between, a "real" and an "imaginary" quantity is called a "complex quantity." Thus $a+ib$, $a-ib$, $ib-a$, &c., are examples of complex quantities.

the thing that exists in thought, has an equal right to be considered real as the thing that exists in the physical world. Indeed, philosophers of the idealistic school go even a step further than this and maintain that only those things are real whose existence is ideal—that the only real table, for example, is the table which exists in mind, *i.e.*, the sum of those sensations which go to make up what we call a percept of table, and that to speak of an external, material table, a table which exists without being perceived, but which is also the cause of our percepts of table, is to talk nonsense. And the idealistic philosopher manages to make out a fairly strong case for his point of view. We do not, however, propose to go to this length; for reasons which we have elsewhere stated we believe in the reality of both the physical and the ideal, or the material and spiritual planes, though we are so far in agreement with the idealistic theory in regarding the reality of the physical world to consist in this: that the physical world exists ideally in the Mind of God.[4] Now, although the mathematician can form no mental image of the

[4] See *Matter, Spirit, and the Cosmos*, Chapter IV, "On Matter and Spirit," and also Chapter VIII, "On Creation."

quantity represented by $i$, he can, nevertheless, deal with this quantity quite rationally—the mathematician can think about this quantity, even if only in a symbolic manner. From our present standpoint, therefore, we must regard both so-called "real" and "imaginary" quantities as real; but whilst the "real" quantities are real in a physical sense (they also, of course, exist ideally in the mind), the reality of "imaginary" quantities is purely ideal—that is to say they represent real thoughts rather than real things, limiting the application of this latter term, for the moment, to the physical plane.

**The Utility of "Imaginary" Quantities.**

§ 42. Probably, however, for many minds, the most convincing proof of the "reality" of "imaginary" quantities will be found in their practical utility; many important results have been obtained in the domain of physical science by the aid of mathematical theorems which have been arrived at by the use of "imaginary" quantities. As an example, we may instance De Moivre's Theorem, which may be expressed by the following formula [5]:

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

[5] To be quite accurate we should say that $\cos n\theta + i \sin n\theta$ is *one* of the possible values of $(\cos \theta + i \sin \theta)^n$.

This formula will doubtless be unintelligible to the non-mathematical reader, but he will notice the occurrence of "imaginary" quantities in it, and he will to some extent realise the importance of these quantities when he learns that this theorem lies at the root of practically the whole of the higher branches of the science of Trigonometry.

**The Significance of "Imaginary" Quantities.**

§ 43. When we consider the matter, it seems a remarkable and significant fact that man has discovered quantities and is able intelligently to employ symbols which are physically unintelligible, which on the plane of the physical are nonsense, but which are seen to be transcendental nonsense, if one may say so—nonsense which results in very good sense. We say that this fact is remarkable and significant, for the reason that it certainly indicates that man is not the mere physical machine that the materialistic philosophers would have us believe him to be. If the mind can deal satisfactorily and rationally with ideas which transcend the physical realm, ideas of undoubted value and utility even when judged by their material results, then clearly must also the mind transcend the domain of the physical.

The fact that we may regard "imaginary" quantities as being equally as real as those quantities commonly so-called, suggests the speculation of a world in which "imaginary" quantities are the common quantities—the "real" numbers, so to speak—that is to say, a world which can be regarded as the embodiment of "imaginary" quantities in a like manner that this world may be regarded as the embodiment of "real" numbers. Now, in our world, the discovery of "imaginary" quantities occurred comparatively late in the history of thought; and because of its transcendental nature there is a difficulty in satisfactorily defining $i$ in terms of "real" numbers. But in the world which we have suggested above, the corresponding discovery, the converse of this, namely, the discovery, starting with "imaginary" quantities, of "real" numbers, would be exceedingly simple, since 1 equals the product of $i$ to four terms, *i.e.*, $i \times i \times i \times i$, or $i^4$, as we have seen above (§ 40). There is no difficulty in defining a "real" number in terms of "imaginary" quantities. This appears to indicate that if such a world exists—a world in which "imaginary" quantities find embodiment—it must be, at least in certain respects, a superior world to ours.

And if we admit that "imaginary" quantities are in any sense real, it seems difficult to avoid the conclusion that such a world must exist. Now, "imaginary" quantities are real in an ideal sense, therefore the world in which "imaginary" quantities are real, the world of which they may be regarded as symbolical, is the ideal, that is, the spiritual world.

**A Mathematical Theory of Spirit.**

**§ 44.** Stated formally, this is our "Mathematical Theory of Spirit": *that just as "real" numbers may be used symbolically to express the various things of the physical world, so, in a similar manner, "imaginary" quantities may be used symbolically to express the various things of the metaphysical or spiritual world.* Such a theory, if valid, supplies a mathematical organon of thought whereby we may pass from the physical to the spiritual realm, and employ mathematical methods in the elucidation of metaphysical problems. One implicate of the theory in question is to the effect that if some "real" number be regarded as representing a certain material thing, then the corresponding "imaginary" quantity, *i.e.*, the product of this real number and *i*, may be regarded as representing the ideal prototype of this thing. In the next chapter we shall attempt to show, along the lines of this theory, how imaginary quantities may be employed in the solution of certain of the problems of Metaphysics.

## NOTE ON THE GEOMETRICAL REPRESENTATION OF "IMAGINARY" QUANTITIES

There is a method of representing "imaginary" quantities geometrically which is of much importance and interest. We give below a brief account of this method; for further information regarding the same we refer our readers to Prof. Loney's *Plane Trigonometry*, Part II, a work to which we acknowledge our indebtedness.

We have pointed out already (*see* § 36 and footnote) that mathematicians have adopted the convention of regarding distances measured to the right of the zero-point as positive, those measured in the opposite direction as negative. Thus in Fig. 9, if the distance OX be represented by $a$, $-a$ will represent a distance equal in length to OX but drawn in the opposite direction, such as the distance OX′. We see, therefore, that the effect of prefixing the minus sign to $a$ is to rotate the line OX through two right angles (say in an anti-clockwise direction).

Now $i \times i = -1$; hence, regarding $i$ as representing some operation, the effect of this operation performed twice is equivalent to the effect of the operation represented by $-1$. But as we have seen above, the effect of the operation $-1$ performed on $a$ is to rotate the line OX through *two* right angles. We may regard the effect of the operation $i$ performed on $a$, therefore, to be the rotation

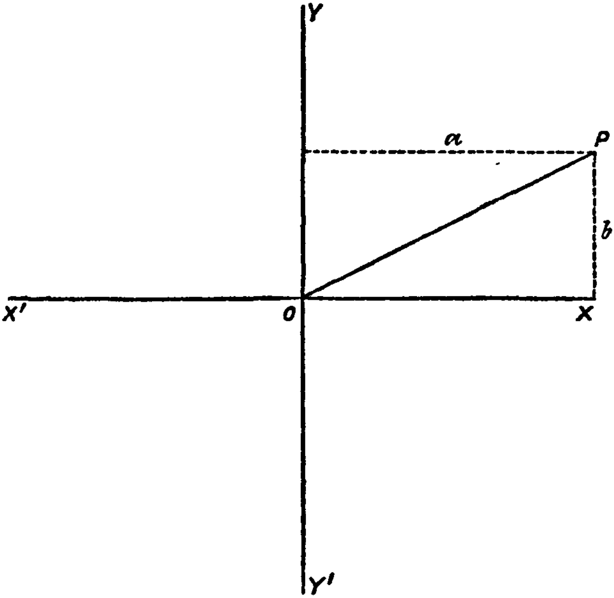


FIG. 9

of the line OX through *one* right angle. Hence, according to this convention, $ai$ represents a line equal in length to the line represented by $a$, but drawn at right angles to it upwards (OY in Fig. 9), whilst $-ai$ represents a line of equal length but opposite in direction to the line represented by $ai$ (OY′ in Fig. 9).

Lines drawn at various inclinations to these axial lines will, following out the above convention, be represented by *complex quantities*.[6] Thus, in Fig. 9, if we suppose that the distance of the point P to the right of the vertical axis is $a$ units, and that its distance above the horizontal axis is $b$ units, then the line OP is represented by the complex quantity $a+bi$. The actual *length* of the line, however, is $\sqrt{a^2+b^2}$, as is evident from the theorem of Pythagoras.

The above method of representing "imaginary" quantities can be employed in demonstration of certain of the theorems connected with such quantities; but the reader should bear in mind its purely arbitrary and conventional nature, and that the fact of these being possible such a method in no wise invalidates the statement that "imaginary" quantities have no real physical significance.

[6] See footnote 3, p. 101.

# CHAPTER VI

## ON THE MATHEMATICAL SOLUTION OF SOME METAPHYSICAL PROBLEMS

**The Correspondence between Matter and Spirit.**

**§ 45.** FROM the standpoint of the theory laid down in the preceding chapter, that is to say, employing "imaginary quantities" as affording an organon of thought whereby we may mentally pass from the realm of matter to that of spirit, we propose to discuss in the present chapter a few important metaphysical problems. The fact that the conclusions arrived at are in remarkable agreement with the views briefly set forth in Chapter I, which are based upon an entirely different line of reasoning, cannot be regarded as otherwise than remarkable.[1]

[1] It is of much interest to note that Swedenborg himself was aware that his Law of Correspondences was capable of being expressed mathematically. In his *Hieroglyphic Key to Natural and Spiritual Mysteries* (Wilkinson's translation, pp. 11 and 12), he says: "Of all the examples we have now brought forward, we may form a kind of proportion or analogy, and of the various analogies, an equation, which latter may be

The first point that we notice is the exact correspondence between the two series, the one consisting of the "real" numbers, the other consisting of the "imaginary" quantities. For every "real" quantity whatever there is a corresponding "imaginary" quantity, namely, the product of $i$ and the "real" quantity in question; and, conversely, for every "imaginary" quantity there is a corresponding "real" quantity, namely, the quotient obtained by dividing this "imaginary" quantity by $i$. Thus 1 and $i$, 2 and $2i$, 3 and $3i$, and, in general, $x$ and $xi$, are corresponding members of these two series. This may be interpreted as being a symbolical expression of the fact that there is, likewise, an exact correspondence between the sum of the material or physical things that make up the physical world and the sum of the

a second time reduced to its constituent analogies. Thus: As the world to man, so are natural effects to rational actions (or in any of the other cases); and if the world be denoted by $w$, man by $m$, effects by $e$, and action by $a$, these signs may be combined in the way of analysis, as $w$, $m$, $e$, $a$ [$w : m = e : a$]. But the proper way to associate them with other signs, and to multiply them, and form an analytic equation, must be explained elsewhere. These will be the first rudiments of that universal mathematics, of which such frequent mention has been made." Swedenborg, however, does not appear to have followed up this idea and worked it out more fully.

spiritual or ideal things that constitute the spiritual world, that for everything material there is some corresponding thing spiritual, and for everything spiritual some corresponding thing material.

**The Distinction between Matter and Spirit.**

**§ 46.** The next point that we notice is the absolute distinctness or "discreteness" between these two series of quantities. Nowhere do they merge one into the other. This point is well brought out in the geometrical representation of "imaginary" quantities. Referring to Fig. 9 (p. 109) we see that the axes XOX′, YOY′ meet nowhere save at the point O, *zero* or nothing. This symbolises the fact that the two worlds of matter and spirit are perfectly distinct or "discrete" from one another. Nowhere do they touch, nowhere do they merge one into the other. It follows also, therefore, as argued in Chapter I (*see* § 5), that spirit must not be regarded (as seems commonly to be the case) as a sort of attenuated form of matter—matter deprived of its substance—nor must matter be thought of as a gross form of spirit.

The increment or decrement in magnitude, quantity or intensity of any property of a material thing must, owing to the discontinuity of matter, proceed by definite steps, and can be represented mathematically by the process of addition or subtraction, as the case may be. Now, by no process of addition or subtraction whatever can we pass from "real" numbers to "imaginary" quantities, or reversely; for by the addition and subtraction of "real" numbers only "real" numbers are obtained, and the addition and subtraction of "imaginary" quantities produces only "imaginary" quantities. This, again, emphasises the point that matter and spirit are not continuous with one another, that the difference between them is neither of magnitude, nor of quantity, nor of intensity, but of some entirely different nature.

**The Origin of Matter.**

§ 47. Now, when we ask, By what process, if any, can we pass from "real" numbers to "imaginary" quantities and reversely? the fact is brought out that by no arithmetical process whatever can we actually pass from "real" numbers to "imaginary" quantities. Combine them how we will, "real" numbers produce only "real" numbers. It was, indeed, only the powers of man's mind that bridged the gulf between "real" and "imaginary" quantities; by purely arithmetical processes "imaginary" quantities would never have been discovered. On the other hand, to pass from "imaginary" quantities to real numbers is perfectly simple, since the fourth power of any "imaginary" quantity is a positive "real" number, *i.e.*, by multiplying any "imaginary" quantity by itself to four factors, a positive "real" number is always obtained. For, since

$$i^4 = 1$$

$$\text{therefore } (ai)^4 = \quad a^4 \times i^4 = a^4 \times 1 = a^4$$
$$\text{and, also, } (-ai)^4 = (-a)^4 \times i^4 = a^4 \times 1 = a^4.$$

Thus :

$$(3i)^4 = 3^4 = 81$$
$$\text{and, also, } (-3i)^4 = 3^4 = 81, \text{ \&c.}$$

In other words, a "real" number cannot generate an "imaginary" quantity, whilst an "imaginary" quantity can generate a "real" number. This may be regarded as symbolically expressing the fact that spirit is not generated or caused by matter, but rather that the origin of the world of matter is to be sought in the realm of the spiritual. It should be noticed, however, that by the above method, the "real" number generated by an "imaginary" quantity (save in the case of $i$ itself) is not the corresponding "real" number. This number can be obtained, however, by multiplying any imaginary by $i^3$, or $-i$. For:

$$ai \times i^3 = ai^4 = a \times 1 = a.$$

Thus:

$$3i \times i^3 = 3i^4 = 3 \times 1 = 3.$$

Following up the analogy, we may interpret this as implying that the causation of the material world by the spiritual world may not only be predicated of the wholes, but that we may, indeed, regard the various elements of the spiritual world as being the respective causes of their material correspondents. (*See* § 6.)

**The Relativity of Experiential Knowledge.**

§ **48.** It is often maintained that all knowledge is relative. So far, at least, as knowledge of the material world is concerned this is undoubtedly true. The appearance of a thing, as we have pointed out already (§ 9), depends not only upon the thing itself, but also upon the sense-organ by means of which it is perceived. The appearance of a thing, in other words, depends upon the relation between the thing perceived and the sense-organ whereby it is perceived. And the same may be said of spiritual perception. In all cases a wrong point of view inevitably results in a distorted view of things.

We must see whether this relativity of experiential knowledge can be mathematically symbolised.

In Mathematics, the relation that one number bears to another is called the *ratio* of the former number to the latter. *The ratio of one number to another is given by the quotient obtained by dividing the first number by the second.* Thus, the ratio of 6 to 3 is 2, since $6 \div 3 = 2$. This ratio is written $\frac{6}{3}$ or $6:3$, and in general, the ratio of $x$ to $y$ is written $\frac{x}{y}$ or $x:y$.

Now, the ratio between two "real" numbers is itself always a "real" number. So also is the ratio between two "imaginary" quantities always a "real" number; for, clearly:

$$\frac{xi}{yi} = \frac{x}{y}.$$

Thus, for example,

$$\frac{6i}{3i} = \frac{6}{3} = 2$$

Moreover, it follows from this that the ratio between two "real" quantities is equivalent to the ratio between the two corresponding "imaginary" quantities, and conversely, according to the formula

$$x : y = xi : yi$$

Interpreting these facts according to the theory we are following out, we arrive at the following conclusions:

**Interpretation of Appearance on the Material Plane.**

**§ 49.** In the first place, we notice that the appearance of a material thing is represented by the same sort of quantity (viz., a "real" number) as that representing the thing in question, though the two numbers are not necessarily identical.[1] The conclusion we may draw from this is that, although material things may not be exactly the same as they appear, there must be a physically real connection between a material thing and its appearance; so that, in the physical world, appearances are not of a misleading nature so long as they are interpreted by physical science, that is to say, so long as one appearance is co-ordinated with and checked by further appearances so as to eliminate the errors of the individual observer. The appearance is that the sun revolves around the earth. This, we know, is erroneous, but it is not absolutely erroneous. The error arises because of the view-point of the observer, and there is a physical reason for it. If we could get off the earth and view things

[1] This is the case only when the other term of the ratio is unity.

from the standpoint of the sun, the appearance would be that the earth revolves about a stationary sun. To get an absolutely correct view of things we should have to take up a stationary position in space. But in spite of the erroneous appearance of the motion of the sun when viewed from the earth, it was by a study of the apparent motions of the celestial bodies that astronomers were enabled to calculate their true motions. And then the principle of the relativity of motion at once explained the apparent motions of these bodies. So that the apparent revolution of the sun leads to error only when relied upon absolutely and not when explained by means of the principle of the relativity of motion. Other examples of what is here meant might be multiplied almost indefinitely.

**Interpretation of Appearance on the Spiritual Plane.**

**§ 50.** In the second place, we notice that the appearance of what is spiritual is represented by a different sort of quantity from that which represents spirit itself; for, whilst the things of the spiritual world are represented by "imaginary" quantities, the appearances of such things for spiritual senses, symbolised by the ratio between two such quantities, are represented by "real" numbers. From this we may conclude that appearance and reality in the spiritual world are, so to speak, of a different order—that the connection between them is spiritual not physical, since the relations between the quantities symbolising appearance and reality in the spiritual world are represented by "imaginary" quantities. To understand such appearances aright, it is not sufficient merely to eliminate any errors due to the view-point of the percipient, but further to interpret such appearances by the Law of Correspondences.

It also follows that the appearance of spirit for spiritual senses is like that of matter for physical senses; and, further, since $xi : yi = x : y$, and since, if $x$ be the number representing something physical, $xi$ represents its spiritual correspondent, it follows that the appearance of a material object for the senses of the body is the same as the appearance of the corresponding spiritual existence as perceived by the senses of the spirit.

These ideas regarding the nature of spiritual appearances help to explain some of the difficulties connected with clairvoyance and allied phenomena; at least, they explain the apparent materiality of alleged visions of the spiritual; for what is beheld is the outward expression of spirit, which is related to its inward nature only by correspondence, just as, in general, matter and spirit are related; hence the appearance seems of a material nature. Spirit is related to spirit as matter is related to matter; this is the metaphysical truth symbolised by the mathematical formula $xi : yi = x : y$. Could we perceive with our spiritual sight the soul of our intimate friend, the appearance would be as if we were beholding his outward nature with the eyes of flesh, assuming there to be an exact correspondence between the inward and outward man, which is not necessarily the case, since we seem to have the power to cloak our inmost feelings and hide hatred under a smile. Of course, we do not imply that all (or even the majority) of supposed cases of spiritual perception, clairvoyance and seership are what they are stated to be, but no more are we prepared to accept the materialistic explanation of all such phenomena. (*See* §§ 10 and 11.)

§ **51.** Now it is clear that:

$$ai : a = bi : b = ci : c = \quad . \quad . \quad . = i : 1;$$

this is the formula symbolising the Law of Correspondences, expressing the fact that the relation between the material and the spiritual is everywhere constant and the same. Now, the ratio symbolical of this relation is equivalent to *i*, *i.e.*, it is an "imaginary" quantity, and it is, therefore, physically inexplicable. Moreover, as we have seen above, the appearances of things both material and spiritual are represented always by "real" numbers, in other words, all experiential knowledge can be symbolised by "real" numbers. Hence, the relation between matter and spirit is one that transcends experience: the physical senses can perceive only what is itself physical, and are unaffected by spiritual existences, whilst material things make no impression on the senses of the spirit, which perceive only the things that are spiritual. It is because we are, in a manner—considered as incarnated spiritual beings—both spiritual and material, inhabitants of both realms, that of spirit and that of matter, that both forms of knowledge, spiritual and material, are possible to us. But because we are essentially spiritual beings, the scepticism of the idealist is also possible to us, and we may doubt the evidence of our outward senses, though we cannot doubt the evidence of pure reason, which is the working of our powers of spiritual perception.

**Further Implicates of the Law of Correspondences.**

Further, we may note that the uses or functions of a thing depend upon the relations which it bears to the whole of its environment and its various parts, so that "use" can be represented mathematically by a ratio or series of ratios. Now:

$$xi : ai, bi, ci \ldots = x : a, b, c \ldots$$

which is merely a variation of the equation $xi : yi = x : y$. We may suppose that $ai$, $bi$, $ci$ represent the elements of the spiritual environment of $xi$, and $a$, $b$, $c$ those of the physical environment of $x$, where $xi$ symbolises something of the spiritual realm and $x$ its material correspondent. Then, from the equality of the ratios, we conclude that the uses or functions of spiritual existences are similar to the uses or functions of their physical correspondents, assuming that the environment of such corresponding things are themselves in correspondence. (*See* § 7.)

. . . . .

**Conclusion.**

§ **52.** It goes without saying that the results of this chapter are of a somewhat speculative nature. It may be, of course, by some mere chance that the relations between the "imaginary" and "real" quantities of the mathematician are so exactly analogous to the relations between the material and spiritual worlds according to Swedenborg; but the analogy seems almost too close for this to be the case, and surely, "real" and "imaginary" quantities are not entirely lacking in metaphysical significance.

One further thought may be ventured: In connection with the geometrical representation of "imaginary quantities," if distances measured along two axes at right angles represent respectively "real" and "imaginary" quantities and thus crudely symbolise material and spiritual existences, it may be asked what would be symbolised by a third axis at right angles to both these former axes (*i.e.*, perpendicular to the plane of the paper). Is there a realm transcending the spiritual as the spiritual does the material? Swedenborg asserts that there is, and calls it "celestial." The spiritual, in his terminology, is the realm of intelligence and wisdom; the celestial that of affection and love. In our ordinary language we call both realms "spiritual," not distinguishing between the two; it may be, however, at the loss of precision, both of language and thought.

We have very briefly and very generally shown along what lines, according to the theory laid down, "imaginary" quantities may be utilised for the solution of some of the problems of a philosophy of the spiritual. If this theory be a valid one, it is evident that the application of "imaginary" quantities to the problems of Metaphysics is capable of considerable extension and amplification, and of being worked out in greater detail; it remains for the criticisms of our readers to show whether the theory be a valid and useful one or not.

THE END

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