THE QUADRATURE OF THE CIRCLE.

CONTAINING DEMONSTRATIONS OF THE ERRORS OF GEOMETRY
IN FINDING THE APPROXIMATION IN USE,

THE

QUADRATURE OF THE CIRCLE,

AND

PRACTICAL QUESTIONS ON THE QUADRATURE,

APPLIED TO THE ASTRONOMICAL CIRCLES.

With an Appendix.

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PREFACE.

THE present edition of this work is not published for sale, and no copy of it will be sold with the author's consent, *at any price*, as the author claims other privileges than a simple copyright, in respect to the alleged discovery. A small edition only has been printed for free distribution to such practical and scientific men as desire to look into the subject, and will give it a thorough, careful and unprejudiced examination.

This work has been written several years, and now appears almost exactly as originally written for publication. It has been delayed in its appearance from motives of interest, and probably would not have appeared now, but that some circumstances seem to have made it necessary for the author to preserve his identity with the principles advanced. It is now submitted for examination, merely, in order that the truths advanced may receive the scrutiny of other minds, and if they are truths as the author affirms, that they may receive the acknowledgment due to their importance. The author, however, seeks no satisfaction for himself; in that particular he trusts to his own examination and judgment. But if the things herein set forth are true, in order to be acknowledged, they must first be known. And with the view of making them known and having them acknowledged, the author has taken the trouble, and incurred the expense, of collating from a mass of demonstrations and printing, that which is now offered as a demonstration of the quadrature.

If this work should receive any attention at all, it will no doubt excite some discussion and some criticism, and as a bantling of the author's, he will, no doubt, be expected to defend it. But the subject is one, which, if the demonstration be true, will, in the end, defend itself, better than any one can defend it: and if it be not true, then no defense can make it outlive the influence of time and scrutiny. It will not be expected, therefore, that the author should notice the objections of caviling minds, or the criticisms of partial and interested judges, who prove by what they say that they have not understood the subject. Nor will any criticism which reiterates the principles which have been disproved in the work itself be deemed worthy of an answer, or anonymous communications be noticed at all. The author's time is too constantly employed, and too valuable to himself, to allow him to indulge in such unprofitable correspondence or controversy.

The points of difference between the author and the schools, which hold that the Quadrature of the Circle is impossible to be demonstrated, are few and easily stated;—they are principally these:—

First, between straight lines and curved lines there is an essential difference in principle and property which has been entirely overlooked by geometers in their approximation.

Secondly, The circumference of a circle is a line *outside* of the circle thoroughly inclosing it.

Third, The line approximated by geometers is a line *coinciding* with the utmost limit of the area of the circle, and if it could be correctly determined would be exactly equal to the circle, but could not *inclose* it or contain it.

Fourth, By this difference, with their method of approximation, geometers make an error in the sixth decimal place.

Fifth, The circle and the equilateral triangle, in their fractional relations to the square, are opposite to one another in ratio of the squares of their diameters.

Any criticism which meets these points understandingly and fairly, showing that the writer has himself understood the subject, will be entitled to an answer; but anything which does not meet these points cannot be considered as being relevant to the points at issue, and therefore cannot be considered as worthy of a reply.

I am not aware of any errors in the book which can mislead any one in regard to the principle or result intended to be shown by the various calculations in figures. It is quite possible, however, that in transcribing for the press, some errors may have occurred; but if any exist, I think they can be only such as the reader will be able to understand and correct for himself.

The work, such as it is, is commended to the examination and candid judgment of all those who may feel interested in the development of truth.

THE AUTHOR.

THE QUADRATURE OF THE CIRCLE.

INTRODUCTORY REMARKS.

CHAPTER I.

To FIND the exact quadrature of the circle, was, from the first establishment of mathematical science, and for many succeeding centuries, a principal desideratum with the mathematicians of those times; but failing in their efforts to arrive at the certainty, and the importance and value of the discovery being, as is generally supposed, greatly diminished by the very close approximations obtained by modern analysis, it has long since ceased, in a great measure, from being an object of research. Nevertheless, being an elementary truth, and consequently the relations existing between the circle and the square being among the fundamental principles necessary to be considered in mathematical science, particularly in respect to astronomy and navigation, the deficiency of an exact knowledge of the quadrature has never ceased to be a cause of some perplexity, requiring explanation, and necessarily, also, a cause of more or less error in these most important of sciences.

Unfortunately, but very much like other men in similar circumstances, the professors of the schools, unable to demonstrate the truth, and unwilling to acknowledge their deficiency, have thought it necessary to explain away their error, and in doing so, they have taken the opposite ground, and concluded, that what *they have been unable to attain, is unattainable by human intellect.* Hence the college lectures on this subject, published and unpublished, abound in learned speculations and hypotheses, tending to place the quadrature of the circle without the range of demonstrable mathematics.

All geometrical truth whatsoever, in nature, rests on two simple things—the properties of straight lines, and the properties of curved lines,—difference of angle and difference of curve are but modifications of the same principles. Of the properties of straight lines, geometers have long supposed themselves to be perfect masters, but of the properties of curved lines, and their relative value to straight lines, geometers have yet known nothing whatever, except by approximation. To find the quadrature of the circle, is simply to determine the relative value of straight lines and curved lines,—and in view of these facts, to an unprejudiced mind, it sounds equally strange and ridiculous to hear the professors of an exact science, condemn as *useless* the solution of a problem, *which is in itself an elementary truth*, and involves, *to say the least, one half of all the geometrical truth in nature.*

Without any disrespect to the learning and general intelligence of those who have adopted the conclusion that it is impossible to find the exact quadrature, (and it seems to be general among the professors,) I may be allowed honestly to doubt both its truth and its reasonableness. It may be admitted, if it pleases them, that it can never be done by any principles at present known or taught in the schools; but this proves nothing more than a deficiency in knowledge of the principles which govern it. To prove satisfactorily that it can never be demonstrated by *any means*, it is necessary, *first*, to prove that no principles *can be true in nature*, but such as are already known to science. To this, however, no mathematician will for a moment pretend, and for aught *any one can know*, there may be other principles in nature, as yet unknown or untaught, and which are equally true with any that are known, by which, when understood, the demonstration may be made. To reason otherwise, is to assume that we are already acquainted with all the principles which the Creator brought into action in reducing matter to form,—an assumption equally presumptuous and improbable. That an essential and fundamental principle which governs the exact relations of straight lines to curved lines *does exist*, is self-evident, and there can be no good reason for saying that it can never be found: and we have the evidence of abundant experience in other things, to show, that when found, it may be as plain, as simple, and as comprehensible as any truth at present within the range of human intelligence. I have been told of Dr. Bowditch, who to his great learning, added great practical skill, and a long course of practical experience, that he was so well satisfied of the existence of numerous principles as yet unknown to mathematical science, but which might hereafter be available for purposes of demonstration, that he would never give his opinion on any original question from what had been previously written or understood on the subject, without first examining carefully, whatever was advanced that was new. There are other mathematicians, however, of far less note than Dr. Bowditch, who do not hesitate to condemn *without examination*, whatever conflicts with their preconceived *opinions*, and this too in the face of any number of facts which may be presented. For myself, I am content to believe that new discoveries of *principles*, as well as new discoveries of truth, are yet to be made, and among others, that which governs the exact relation of one circle to one square. I cannot understand how *limited* and circumscribed matter is made infinite with respect to other limited and circumscribed matter by the mere influence of shape. It may be true in respect to decimal parts, and consequently in respect to decimal figures in their decimal relation to *one*, but it is not true on the broad principle, and therefore it *is not true* in respect to the circle and the square. While, therefore, I am ready to admit that the quadrature of the circle can never be demonstrated by any means yet adopted for that purpose, I am still compelled to regard the reasoning which places it beyond the reach of demonstration, as containing more theoretic learning and speculation than simple truth, and those who really believe it to be among impossibilities, as more credulous of mystery than sound in their judgment.

By a course of reasoning on the mechanical properties of numbers, I have adopted a ratio of circumference to diameter of *one circle*, which I believe and affirm to be *the true and exact ratio* which nature employs in every circle, and which, therefore, in order to distinguish it from all others, I denominate *the primary ratio* of circumference and diameter. The numbers expressive of the value of each are 20612 parts of circumference to 6561 parts of diameter. Diameter being *one*, these numbers will give a decimal circumference = 3,141594 + which will be seen to be greater by one and more in the sixth decimal place, or about $\frac{1}{3,000,000}$ greater than Playfair and Legendre's so called perimeter of a circumscribed polygon of 6144 sides. Consequently, before I can claim attention to my ratio as likely to be the true one, it is necessary for me to show Playfair and Legendre's liability to an error, equal to the difference. I shall proceed to do this by the selection of a few from a mass of demonstrations which are before me to that effect. In the pursuit of this object, and throughout the following work, my course of reasoning will be *altogether original.* I shall make no quotations, and refer to no authority, for in fact I never consulted an author on this subject, until long after I had substantially finished my demonstrations. Everything, therefore, contained in the reasoning which follows, is, so far as I am concerned, purely original, and every truth illustrated and proved, even to the measuring of a triangle, is as much my own discovery as if it had never been known before. The extent of my acquirements in geometry *at school,* was to learn that the hypothenuse of a right-angled triangle is greater than either side; there I stopped, went to business, and from that time to the present, a period of nearly forty years, I have been the originator of my own system of mathematics, independent of any instruction but such as has been suggested to my own mind by observation of natural truth, and an examination of the powers and properties of numbers. The following work, therefore, is entirely the production of my own mind, reasoning from nature and nature's laws only, and no other mind has, to my knowledge, either contributed or influenced a single truth or idea illustrated in it. On the contrary, I have, with scrupulous care, rejected every thing suggested by any other person, even though the suggestion might have been beneficially improved.

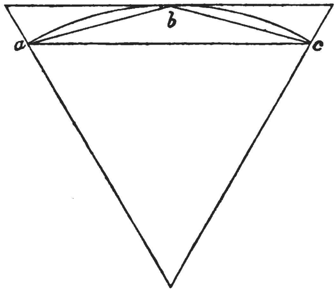
My reasoning, I think, will be found to be perfectly conformable to nature, yet not confined to the rules of art; and as all original reasoning necessarily involves some new truth, or a handling of old truths in a manner not before practiced, I have found it both convenient and necessary to adopt such terms and forms of expression, as, in my judgment, would best convey my own meaning and suit my own purpose, without regard to their common scientific application, preferring always the attainment of truth, rather than elegance. Thus, for example, the term "circumference," when it will best suit the idea intended to be conveyed, is often applied to the square, the triangle, to polygons, and other angular shapes, and is synonymous with perimeter. "Diameter" is also applied to triangles and rectangular figures, and diameter, when not otherwise explained, always means twice the least radius (which, in all regular shapes, is always the diameter of an inscribed circle) when a figure is to be measured by circumference and radius; but in the equilateral triangle when it is to be measured in the usual way of measuring angular shapes from half its side by the perpendicular, then the perpendicular from the base is the diameter. This seeming indiscriminate use of terms will, I think, be better understood by the reading, and will not be objected to except by those who reason for elegance rather than truth, and those, who, for lack of reason, confound terms and definitions with principles.

DEMONSTRATION OF ERROR
IN THE
APPROXIMATION OF GEOMETERS TO THE CIRCUMFERENCE OF A CIRCLE.

PROPOSITION I.

The circumference of a circle is greater than any number whatever of mean proportionals from an inscribed straight line.

PLATE I.



We have here a section of circumference with an inscribed and circumscribed perimeter of a polygon. The circumference itself gives ocular demonstration, that it is greater than any number whatever of mean proportionals, as the straight lines *a b*, or *b c*, from the inscribed straight line *a c;* because, as all such mean proportionals must forever continue to be *inscribed* straight lines, they can therefore never equal the circumference; and without this quality the circumference *could not be a perfect curve. The proposition is therefore demonstrated.*

From the above proposition it is evident, that, if we approach the circumference of the circle by any scale of proportionals whatever between the inscribed and circumscribed lines, other than that which is a true and exact *relative* to the circumference, we shall inevitably bring the circumscribed lines within the circumference, or carry the inscribed lines without the circumference, just in proportion as the scale of proportionals used is greater or less than the true and exact relative to the circumference; and because, as already seen, the circumference is greater than *any number whatever* of mean proportionals from an *inscribed* straight line, therefore any number of mean proportionals which may bring the inscribed and circumscribed lines to agree with each other in whole or in part, will necessarily make them to agree at a point of value *within the value of circumference:* And the approximation being carried to a large number of figures, you will have an approximation within less than one in the last decimal place of a line of figures, *not* *of a full diameter*, but of a diameter which, in its ratio of value to circumference is also less than *one*. I say, it is a diameter which is less than one by $\frac{5+}{100000000}$. And *whatever* may be the scale of proportionals used, if it be any other than the true and *exact relative* to circumference, there must necessarily be a point in a line of figures representing the circumference of one diameter, where the error of the scale used will amount to *one or more*, and that point will be forever the same, by whatever form or process the calculation is carried on.

Here then is a plain *fact*, of which we have ocular demonstration, depending on the plainest laws of numbers and geometry, and which needs only to be admitted, to prove, that the approximations of geometers to the circumference of a circle obtained from inscribed and so-called circumscribed straight lines *is less than the true value of circumference.* In a future demonstration I shall show that the perimeter of the so-called circumscribed polygon of geometers, is *not* a circumscribed polygon, but that at any number of sides the perimeter at the center of each side rests within the area of the circle, and at an infinite number of sides is brought *wholly within* the area of the circle. (See prop. L, Appendix.)

It follows therefore that whenever the true ratio of circumference to diameter is found, it will disagree with, and be greater than the approximate ratio of geometers, and greater than the so-called perimeter of a circumscribed polygon obtained in the manner which it is. It matters not that some have had the patience to carry the geometer's approximation to 50 or even 150 decimal places, the more the worse; they do not approach any nearer to the true circumference by so doing. On the contrary, from the point in a line of figures where the error in the scale of proportions used amounts to one or more, and where they necessarily pass to the inside of the value of circumference, all effort to gain any nearer approach to it, is nothing more than an effort to arrange a vulgar fraction of remainder into a line of decimals, which because it is not of decimal value makes a line of figures without end.

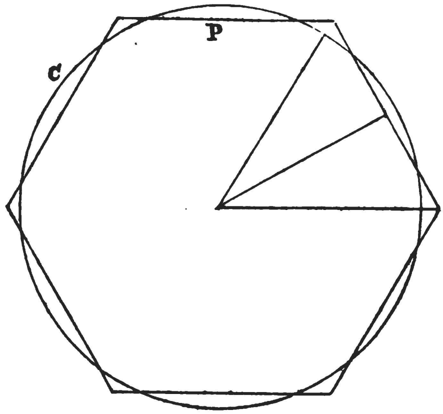
I have stated the case in such general terms as that the error of geometers shall be seen to apply, as it really does, to *every method in use* for finding the approximation by means of straight lines, or by any fluxionary series, the basis of which is the value of straight lines; and in order that we may understand it fully, we will now look further at the mechanical causes which can have this influence. And

First. Between curved lines and straight lines there is an essential difference in principle and properties, as is evident from the fact, that if two shapes be formed, one with straight lines and the other with curved lines, and of equal magnitude or area, their circumferences will be totally different, and again if their circumferences are equal, their areas will be totally different; and this difference of principle or property, is therefore seen to consist chiefly in the different powers of the two descriptions of lines to inclose area; therefore,

PROPOSITION II.

The area of a circle is greater than the area of any shape formed of straight lines and of equal circumference, and hence greater than the area of any polygon of any possible number of sides, and having the same circumference with the circle.

PLATE II.



In the figure (Plate II.), let C be the circumference of a circle, and let P be the perimeter of a polygon, and let C and P be equal to one another in length. It is evident that C is a perfectly curved line, and that P is composed of several straight lines. It is known also that all regular shapes formed of straight lines and equal sides, have their areas equal to half the circumference, or half the length of all its sides, multiplied by the *least* radius which the shape contains, than which all other radii contained in the shape are greater; whereas, the circle has its area equal to half the circumference by *the radius*, to which every other radius contained in the circle is equal. (prop. I., chap. ii.). It is evident that the radius of C is less than the greatest, and greater than the least radius of P. It is evident also, that so long as P shall have the properties of straight lines, to whatever number its sides may be increased, it shall always have a greater and a lesser radius, and its area always equals half the circumference by the *least* radius; while C has never but *one* radius, which is always greater than the least, and less than the greatest radius of P; and since C and P are known to be equal in length, therefore C multiplied by radius, is greater than P multiplied by *least* radius. Therefore while P shall continue to have the properties of straight lines it can never equal C in area, though the number of its sides were infinite, or the greatest possible. But if the number of sides of P shall be infinite or the greatest possible, then P shall approach so nearly in area to C, that the difference *cannot be less.* And in material things, *any difference* such that it cannot be less, is one ultimate particle of whatever material or thing is under consideration. (For ultimate particles of matter see explanation, prop. II., Appendix). *The proposition is therefore demonstrated.*

By proposition I. it has been shown, that the circumference of a circle is greater than anything which can be reached by mean proportionals between inscribed and the so-called circumscribed straight lines at the point of value where the two agree. And by proposition II., it has been shown, that the area of a circle is greater than any area which can be contained within any possible number of straight lines having the same aggregate length of the circumference of the circle. It is evident therefore that in attempting to find the value of the circumference by means of straight lines, we shall always arrive at results which are less than the truth: *and geometers have never used anything else but straight lines.* It is evident, also, that if we reason wholly on the properties of straight lines and the principles which govern *them*, we leave wholly out of the account the properties of curved lines and the principles which govern *them.* And because the two differ essentially from one another, we therefore determine nothing by it of their relative value, further than to arrive at an approximation; and there is still an error *unaccounted for*, which stands *additional to* all the errors arising from quantities neglected or lost in the usual mode of finding an approximation.

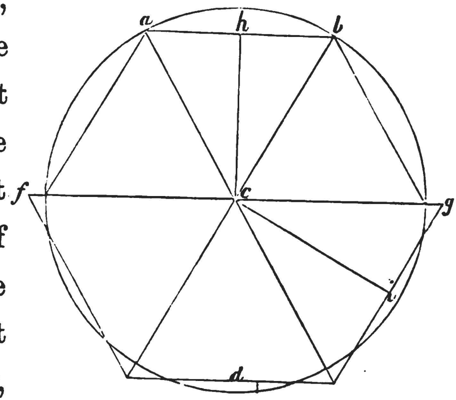
When we look for a mechanical cause which can make the circumference of a circle greater than any number whatever of mean proportionals from an inscribed straight line (as *a b* or *b c* from the inscribed straight line *a c*, prop. I.) we see that nothing is adequate to that purpose, but the infusion into circumference, of some additional value after all mean proportionals have ceased; which is, when the inscribed and circumscribed so-called have become thoroughly equal; and whatever the amount of this infusion may be, it is evident, that there must be a point in the expression of numbers, where its value becomes *fixed*, and where it shall amount to *one* or *more;* and because this infusion into the value of circumference, is a fixed and unalterable law of nature, by which the two descriptions of lines are made essentially to differ, therefore, whenever we attempt to find the value of circumference by the properties of straight lines alone, we shall always fall short of the truth, at that point in the expression of numbers, where the value of this essential difference equals one or more. To get rid of this essential difference of property in the two descriptions of lines (for they are not ignorant of its existence) geometers have assumed that it is "Infinity," and therefore call it nothing! but I shall show, that whatever an infinity may be, it is always such, that in material things it is capable of *increase*, and is therefore a value *not to be thrown away.*

Secondly. The perimeter of a circumscribed polygon of any number of sides, is a proportion to the *inscribed* of an equal number of sides, in precisely the same ratio as the radius is to the perpendicular from the center to the chord of the arch. Now, therefore, if the circumference be greater than *any number whatever* of mean proportionals from the inscribed, as we have ocular and mathematical demonstration in proposition I. that it is, and without which it could not be a perfect curve, then it is evident again, that the circumscribed, is to the inscribed, in *less ratio of value* than the circumference. Here again we see a mechanical necessity of any proportions between the inscribed and the so-called circumscribed lines, other than such as are exact relatives to the circumference, necessarily bringing the two to agree with each other, at a point of value within the value in area of the circumference; therefore

PROPOSITION III.

The perimeter of any polygon whose circumference or the length of whose sides is equal to the circumference of a circle, is always such that if the polygon be placed or drawn upon *the circle, the point or angle formed by the* *hypothenuse of each right angle which the polygon contains, lies wholly* outside *of the circumference of the circle.*

PLATE III.



Let the perimeter, *f*, *d*, *g* (Plate III.), equal the semicircle over which it is drawn, *i. e.*, let the length of the straight lines, *f*, *d*, *g*, equal half the circumference of the circle. It is seen that the point or angle *g*, formed by the hypothenuse *c*, *g*, from the right angle *c*, *i*, *g*, lies wholly *outside* of the circumference of the circle, and although this extension will be greatly diminished by increasing the number of the sides of the polygon, yet so long as the circumference of the circle, and the perimeter of the polygon are equal, to whatever number the sides of the polygon may be increased, the point or angle, *g*, shall always lie *wholly outside* of the circle. *The proposition is therefore demonstrated.*

But it is seen that the hypothenuse *c a*, formed by the right angle *c*, *h*, *a*, lies wholly *within* the circumference of the circle, it (the right angle *c*, *h*, *a*,) being part of an *inscribed* polygon; and it is evident, that so long as it shall remain an inscribed polygon, to whatever number its sides may be increased, the angle of hypothenuse, or point *a*, can never extend beyond the circumference of the circle. It is self-evident, therefore, that any scale of proportionals which shall bring the sides of an inscribed and circumscribed polygon to agree with each other after the manner of geometers, within less than any assignable quantity, and which does *not* carry the angle of hypothenuse outside of the circumference (which the method of geometers does *not do*), is less than a true and exact relative to the circumference, and it is equally evident that the circumference of a circle is in greater proportion to the inscribed perimeter, than such inscribed perimeter is to the perpendicular on the chord of the arch; and because the inscribed and circumscribed, so called, are in proportion to one another as the perpendicular on the chord of the arch is to radius, therefore, the two are brought to agree with each other at a point of value less than the circumference of the circle. It will be seen, also, from the illustration (Plate III), that if the sides of any polygon shall be equal to the circumference of a circle, the polygon shall have a diameter greater than any inscribed polygon, in the proportion as *c*, *i*, or *c*, *g*, is to *c*, *h*, or *c*, *a*, and to whatever number of sides the polygon may be carried, whether 6000, or 6,000,000, the sensible difference of diameter will still be in the same proportion as *c*, *i*, or *c*, *g*, is to *c*, *h*, or *c*, *a*, in a polygon of six sides. This difference will be greatly diminished by any increase of the number of sides, but can never be exhausted, and hence, as stated in the demonstration to my first proposition, any approximation found by geometer's methods, by proportions between the inscribed and so-called circumscribed perimeter of a polygon, which brings the two to agree with each other within less than any assignable quantity, and which *does not* carry the angle of hypothenuse *outside* the circumference, is an approximation *not* of a full circumference, but of a circumference whose *diameter is less than one*, in its ratio of value to the area of the circle: therefore—

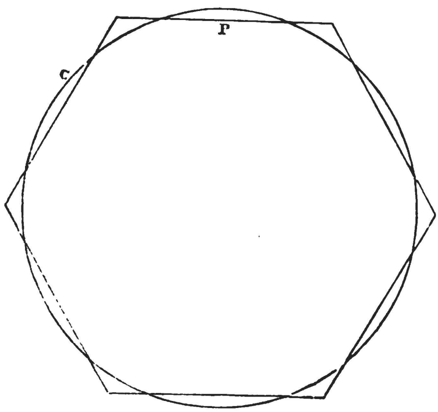
PROPOSITION IV.

If the perimeter of any polygon be brought into the form of a circle, both the diameter and area of the circle are greater than the diameter and area of the polygon.

By diameter of a polygon, I mean always its least diameter, or that which, being multiplied by one-fourth of the perimeter, or its half multiplied by half the perimeter, gives its area, which is always the diameter of an inscribed circle.

By proposition II., it is shown that if P be the perimeter of a polygon, and C be the circumference of a circle, and P and C are equal to one another in length, then P shall always have a greater and a lesser radius, and the

PLATE IV.



radius of C shall always be greater than the least, and less than the greatest radius of P, therefore, C multiplied by radius is greater than P multiplied by least radius. It is evident, therefore, as seen in Plate IV., that if the perimeter of any polygon be brought into the form of a circle, both diameter and area are increased by the transition of shape. *The proposition is therefore demonstrated.*

Now, it is known that geometers regard a many-sided polygon as *equal* to a circle, to the extent to which two polygons (the inscribed and circumscribed, so called) agree with each other; and with them, to the same extent, diameter is treated as having a *fixed* and equal *value*, whether the shape may be a many-sided polygon or a circle. Let us then examine the effect of considering the two shapes as equal to one another, and each having the same *fixed diameter*. Let P (Plate IV.), be the perimeter of a polygon, and let C be the circumference of a circle, and let C and P be equal to one another in length. It is already known that the area inclosed by C is much greater than the area inclosed by P. If, therefore, the diameter of each, C and P, be considered as *fixed*, and treated as *one*, then, in order to give such expression to the circumference of the circle as that its half being multiplied by $\frac{5}{10}$, or half the diameter, the result shall express the whole area of the circle, we must add to the length of C an amount equal to *four times* the difference between the area inclosed by P and the area inclosed by C, which would make C to be much greater than itself, and the idea is absurd. But it is self-evident, that what is here so manifestly true in the transition of a perimeter of a polygon of six sides to the circumference of a circle, is equally true, though in less proportion, in the transition of shape of a polygon of 6000 sides, or of any other possible number, and to regard them as equal is just as absurd in one case as in the other. Yet this effect of the transition of shape is entirely disregarded by the geometry of the schools in finding their approximation to the value of circumference.

The difference between a polygon of some thousands of sides, and the circumference of a circle of only one or two inches in diameter, like the common diagrams in use, is a difference inconceivably small. But in order to realize the importance of the principle contained in the transition of one shape to the other, it is necessary for us to enlarge our ideas of magnitude as we increase the number of sides of the polygon. Therefore, let the earth's orbit be a polygon of 6000 sides, then each side will be a straight line of more than ninety thousand miles in length—the difference between the least radius of the polygon and the radius of a circle having the same circumference will be ten miles or more—the mean will be, say five miles, and as but a small portion of the area of the polygon lies outside of the circle, the difference between the area of the circle and the area of the polygon will be nearly equal to a belt, say four to five miles wide, and five hundred and eighty millions of miles in length, an area several times greater than the whole surface of our globe; and if the number of the sides of the polygon be increased a thousand times, the difference of area will still equal an empire in extent. Again, if we take a radius equal to the distance of one of the fixed stars, and again increase the number of sides of a polygon by millions, yet there, the difference of area between a polygon, and a circle having *the same circumference*, will again exceed the whole surface of our globe a thousand times. Yet this is that *infinity* arising from the transition of shape, and *increased* with the increase of magnitude, which (as I have said in proposition II.) the geometry of the schools regards as *nothing*, and therefore throws it away!! and this, also, is that infinity, which, in solving the problem of the circle, we must grapple with,—grasp it, and hold it fast within our comprehension and control, so that nothing can escape our notice, which is capable either of increase or diminution.

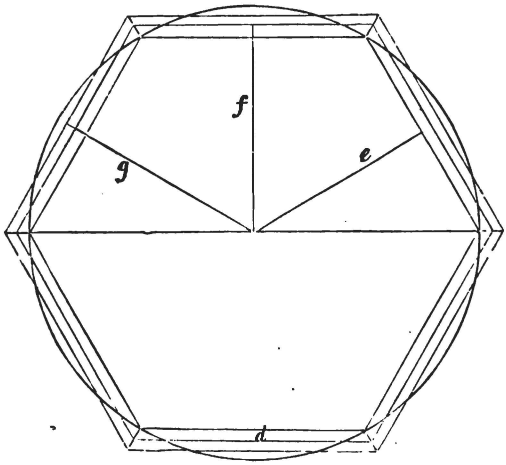
By proposition II., I have shown that the area of a circle is greater than the area of any polygon having the same circumference, or length of perimeter with the circle. Therefore

PROPOSITION V.

If the area of any polygon shall equal the area of a circle, the perimeter of such polygon shall lie wholly outside of, and inclose a polygon, whose circumference equals the circumference of the circle.

This proposition is already proved by the demonstration of proposition II., and needs not to be repeated, because, if the area of a circle is greater than the area of a polygon having the same circumference as the circle, then by reciprocity, if the *area* of a polygon shall *equal the area* of the circle, then the circumference of such polygon shall be greater than the circumference of the circle, and shall be able to inclose another polygon, whose circumference *equals* the circumference of the circle.

PLATE V.



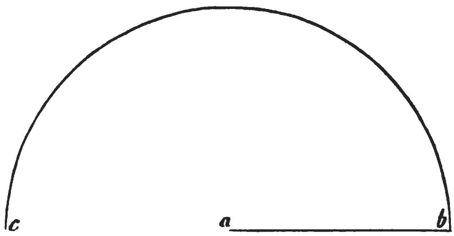
We have then, in Plate V., three polygons, one whose perimeter rests on the perpendicular *e*, and which differs from an inscribed polygon less than any assignable quantity, and which, therefore, equals the approximation of geometers, at six sides of a circumference. Another, whose perimeter (*d*) rests on the perpendicular, *f*, and which *equals the circumference* of the circle; and still another, whose perimeter rests on the perpendicular, *g*, and which incloses an area *equal to the area* of the circle. Between these three polygons, at six sides of a circumference, there is a very sensible difference of *area* and *diameter*, and to whatever number of sides these polygons may be bisected, they can never be brought to agree with each other, but there will always remain a positive difference of area, and a difference of diameter, which is more than three times equalled in the circumference. It is known, and I have proved (in proposition XII., chapter ii.), that the true ratio of circumference to diameter of all circles, is four times the *area* of one circle inscribed in one square for the ratio of circumference, to the area of the circumscribed square for the ratio of diameter; and since it is proved that the area of a circle is greater than the area of any polygon of any possible number of sides, and having the same circumference with the circle, it is therefore evident, that if we would find the ratio of circumference which shall express the full value of the area of the circle by proportions between the perimeters of an inscribed and circumscribed polygon, we must seek it between the inscribed, and one circumscribed, which stands to the square of radius, in the proportion as the square of the perpendicular, g (Plate V.), stands to the square of perpendicular, f, and unless we do this, we leave wholly out of the account that essential difference between the properties of straight lines and curved lines, which enables the latter to inclose more area than any possible number of straight lines in any shape, and of the same aggregate length with the circumference of the circle. It is evident, that between two such polygons there will be a point in the expression of numbers, within which they can never be brought to agree with the inscribed, but for bisecting two such polygons, or for finding such a circumscribed polygon, the analysis of geometers affords not the slightest means; and by their method, they aim only at the value in area of the perimeter resting on perpendicular, *f*, which is less than the area of the circle by the essential difference in the properties of straight lines and curved lines. It must not be forgotten, that this error of principle in geometer's methods of finding an approximation, is *additional* to all the errors arising from quantities neglected or lost in the calculation, which being errors of each side of the polygon, are, by the increase of the number of sides, made of great aggregate value in the circumference; and what is here so manifestly true, at six sides of a circumference, is equally true, but in less proportion, at any possible number of sides; how much more then is it true at a few thousand sides, to which only geometers have carried the bisection.

These remarks and illustrations might be extended almost without limit, but if I am understood in what I have already said, I think it quite sufficient to show, that the analysis of geometers of the value of curved lines, and the equality of the area of a circle to a square is erroneous in principle, and therefore without merit, except as a near approximation, and in the absence of any exact knowledge of the truth. It will be equally certain, also, that the true circumference, when found, will be greater than the ratio of geometers at no very remote point from unit. Whether it is as much greater as I have affirmed it to be, remains to be shown by the direct propositions which shall prove it.

I had supposed until recently that I stood alone in my views respecting the quadrature, but in looking over Playfair's Notes on his Supplement to the Elements of Geometry, we are there informed, that Torrelli, a learned Professor of Oxford, who had then recently published an edition of the works of Archimedes with comments, had taken the ground that by their modern analysis, geometers prove nothing whatever respecting the properties of curved lines; which is precisely the thing I have here shown.

If all I have said be true, then I have effectually and *mechanically* shown the utter impossibility of ever squaring the circle by the application of straight lines, or by any method now in use, and I am as completely satisfied on that point, as the skepticism of any modern Professor respecting my solution of the problem could desire. But this does not prove that the circle is incapable of being squared, or that no equality exists between the circle and the square, which I shall presently show by direct propositions *does exist;* and unless it can be shown on the other side, that no principles in geometry can be true but such as are already known, I have yet hopes of arriving at the quadrature. That whenever it is reached it will be greater than the approximation of geometers, and greater at no very remote point from unit, than the so-called perimeter of a circumscribed polygon which is made to agree with the inscribed within less than any assignable quantity, I think already established; if not, I think additional facts will not be wanting to place my position beyond a doubt.

PLATE VI.



The value of the area of a circle, is the value of a straight line *a b* (Plate VI.), equal half the diameter, multiplied by the value of the curved line *c b*, equal half the circumference. The value of the curved line *c b* is, as we have seen, greater than the same length of straight line at any possible number of sides, and by my ratio of circumference to diameter, which I affirm to be the true one, the expression of numbers by which circumference and diameter are made equal, is 20612 parts of circumference to 6561 parts of diameter, or *a b*=3280.5 and *c b*=10306, which I now propose to prove.

CHAPTER II.

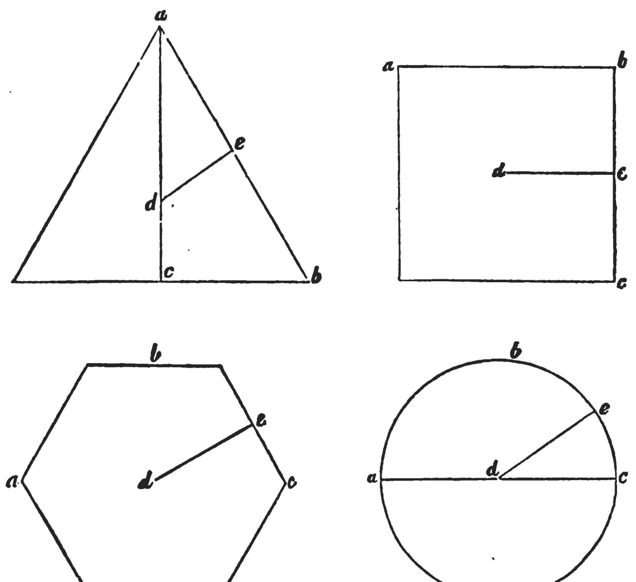
THE QUADRATURE OF THE CIRCLE DEMONSTRATED.

In the attempt made in the preceding chapter to show the existence of fundamental errors in the analysis of geometers, we have seen, very clearly, I think, that there is an essential difference in the properties of straight lines and curved lines, which has been entirely overlooked in the approximations heretofore made; but the knowledge gained by these examinations thus far, is only of a negative character, showing us only what is *not true* respecting the properties of curved lines, and thus disabusing our minds of a preexisting error. We are now by a course of direct propositions to inquire and determine what *is true:* and in this inquiry the *relative properties* of straight lines and curved lines demand our first attention, that we may thereby be enabled to discover their *relative value;* therefore,

PROPOSITION I.

One of the relative properties between straight lines and a perfect curve or circle is such, that all regular *shapes formed of straight lines and equal sides, have their areas equal to half the circumference multiplied by the* least *radius which the shape contains (which is always the radius of an inscribed circle) than which every other radius contained in the shape is* greater, *and the circle has* its area *equal to half the circumference multiplied by* the radius, *to which every other radius contained ia the circle is* equal.

PLATE VII.



It is not necessary formally to demonstrate the above proposition to any mathematician. I will therefore only state that in the above figures (Plate VII.) the area of the equilateral triangle equals the lines *a b c* × *d e*, or half the circumference by the least radius which the triangle contains, which is always the radius of an inscribed circle, and it is evident that every other radius contained in the triangle is greater than the least. In like manner the areas of the square and the hexagon are equal to the lines *a b c* (half the circumference) by the line *d e* which is the least radius either shape contains, than which every other radius contained in either shape is greater. But the area of the circle equals the line *a b c*, half the circumference, by the radius *d e*, to which every other radius contained in the circle *is equal. The proposition is therefore demonstrated.*

Here then is a relative property between straight lines and curved lines, showing us conclusively, that since straight lines have been made the basis of area in mathematical science, some compensation must be made to the circumference of the circle for this difference of *relative property*, if we would give to the circle the full expression of its value, by the properties of straight lines, *which is the thing demanded by the quadrature;* and a future demonstration will show what this compensation shall be.

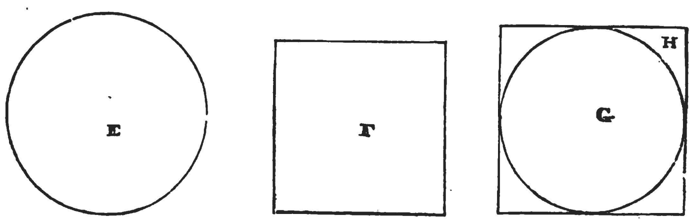
The next relative property of straight lines and curved lines which I shall notice is contained in the following proposition.

PROPOSITION II.

The circumference of any circle being given, if that circumference be brought into the form of a square, the area of that square is equal to the area of another circle, the circumscribed square of which is equal in area to the area of the circle whose circumference is first given.

EXPLANATION.—In the figures E, F, G and H, Plate VIII., let the circumference of the circle E be given,—let it be for example 36 (or any other number), and let the circumference or four sides of the square F be also 36, then one side of F=9, and 9 × 9=81, which is the area of F. Now let the area of the circle G=81, then by the proposition the area of the square H circumscribing G equals the area of the circle E whose circumference is 36.

PLATE VIII.



My own mode of demonstrating the foregoing proposition is simply to test the principle by more than one ratio of circumference and diameter. If it be true of these, then it is a general principle, and true of every ratio of circumference and diameter which can be used, which is the fact of all general principles applied to the properties of the circle.

Therefore, let the circumference of the circle E equal C; then if the geometrical approximation be taken as *the ratio* of circumference, and the diameter of E=1, then the circumference of E (C) will equal 3.1415926+ and the area of E will equal .7853981+. The area of F will equal $\frac{C^2}{16}$ and the area of G being equal to the area of F, therefore the area of G=$\frac{C^2}{16}$, and if G=$\frac{C^2}{16}$, then the area of H equals .7853981+which is also the area of E; and the diameter of G=$\sqrt{.7853981+}$. But if *my ratio* be taken as *the ratio* of circumference to diameter, and the diameter of E=1, then C=3.1415942+, and the area of E=.7853985+: the area of F=$\frac{C^2}{16}$, and if the area of G=$\frac{C^2}{16}$ then the area of H=.7853985+ which is also the area of E, and the diameter of G=$\sqrt{.7853985+}$. The proposition is therefore demonstrated to be a general principle which is true of every ratio of circumference to diameter, and is therefore necessarily true of *the true ratio.* But being a general principle and wholly independent of any particular ratio of circumference and diameter, this proposition does not of itself demonstrate what *the true ratio* of circumference to diameter *is*, nor is it in the power of any general principle to do so. Another demonstration will show that the circumference and diameter of one circle is the basis of all general principles relating to the extension of area, and, therefore, the relation of circumference to diameter is of itself, a *particular fact*, depending on *particular facts*, and not on general principles of which it is itself the basis. Yet numberless arithmetical and geometrical problems may be based on this proposition, which are entirely new in mathematics, perfectly simple, and may be made in the highest degree useful.

One point, however, and that an essential point, and the first point necessary to be shown, has been gained by this last proposition, viz., the existence of a perfect equality between the circle and the square has been clearly shown. For if the circle and the square, or what is the same thing, if the circumference and diameter of a circle be really *incommensurable* as geometers have affirmed, then no circle and square *can* be exactly equal one to the other. But when it has been demonstrated as has here been done, that a circle and a square may be exactly equal one to the other, then it is demonstrated also, that the two are *not incommensurable;* and with this demonstration the whole theory of mathematicians respecting the non-existence of any expression of numbers by which the circle and the square are made equal is proved to be fallacious.

The inverse of proposition II. will read thus: Any square (H) is such, that its inscribed circle (G) is equal in area to the area of another square (F) whose circumference equals the circumference of another circle (E) of equal area with H. And in this view of the proposition let it be particularly noticed, that nothing is assumed, and the result is subject to no conditions as in the direct proposition, which is conditioned on the equality of G to F.

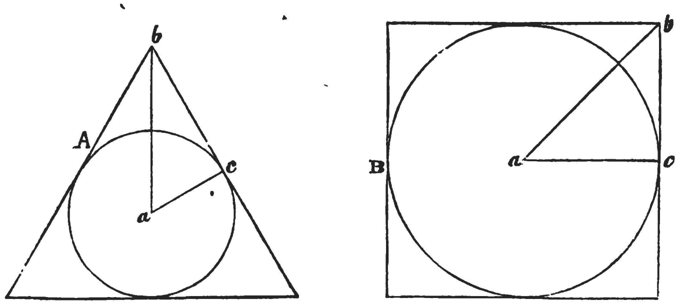
This 2d proposition is not only wholly original *with me*, but is, I believe, entirely new in mathematics; and certainly it is very beautiful as showing the principles and the manner in which the areas of circles and squares unfold and display to each other. It was discovered by me 6 or 7 years since by the method of demonstration here given, and it has since been frequently demonstrated by algebraic formula; but I forbear to insert any of these demonstrations and give as a reason for this omission,—*first*, that I never make use of algebra in demonstration, and *secondly*, that, although the principles of algebra aided by geometry are amply capable of demonstrating the proposition when *discovered and stated*, it yet to my mind embodies no principles or process of reasoning by which the discovery *can be made*. It is not in the power, I think, of algebra alone, by the same formula, and without the aid of numbers or geometry, to contemplate the transition and alternation of shapes necessary for the discovery of this proposition. And hence I attribute to the almost exclusive use of algebra in the schools, and the little attention paid to the mechanical properties of numbers, that this relative property of the circle and the square, or of straight lines and curved lines, has so long remained unnoticed and undiscovered.

The following proposition is, I think, appropriate here to the course of my reasoning.

PROPOSITION III.

The circle is the natural basis or beginning of all area, and the square being made so in mathematical science, is artificial and arbitrary.

PLATE IX.



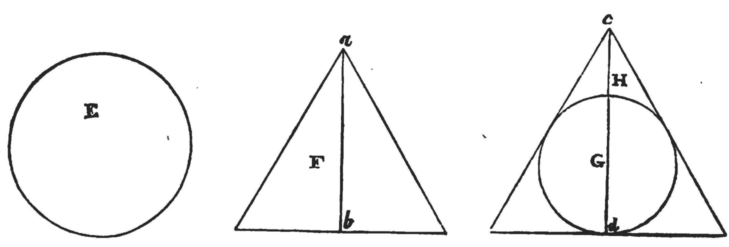
By proposition I. it has been shown, that all regular shapes have their areas equal to half the circumference by the radius of an inscribed circle, which is the least radius the shape contains; the circle is therefore the basis of area in all such shapes. If we examine the figures A and B (Plate IX.) we see that the square and the triangle have each a greater and a lesser radius, which is true of all other shapes formed of straight lines. We see also that the circle described on the least radius of either shape is less than the shape. We see also that neither shape can be so diminished, but that a circle described upon its least radius, will be less than it. Therefore if either shape shall be diminished to infinity, so long as it shall have magnitude or area, it shall also have a greater and a lesser radius, and a circle described upon its least radius will be less than it. The circle is therefore the least of all possible magnitudes, and, as all extension must be from the least possible magnitude to that which is greater, therefore the circle is the beginning of plane extension, and is hence the natural basis or beginning of all magnitude or area, and the square being made so in mathematical science is artificial and arbitrary. *The proposition is therefore demonstrated.*

Now, therefore, because the circle is the *natural* basis of *all area*, and the square is made the artificial basis, and because proposition II. is a relative property between the circle and the square, and between straight lines and curved lines, and is also a general principle applicable to all ratios of numbers; and because the square is a regular shape formed of perfect straight lines and of equal sides and angles, therefore proposition II. is a relative property between the circle and *all regular shapes* whatsover, which are formed of straight lines and of equal sides and angles. Therefore,

PROPOSITION IV.

The circumference of any circle being given, if that circumference be brought into any other shape *formed of straight lines and of equal sides and angles, the area of that shape is equal to the area of another circle, which circle being circumscribed by another and* similar shape, *the area of such shape circumscribing the last named circle is equal to the area of the circle whose circumference is given.*

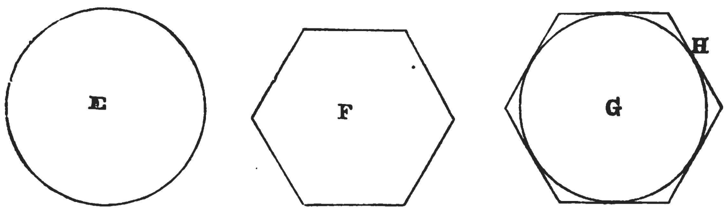
PLATE X.



Let the circumference of the circle E (Plate X.) equal 3, then the area of E (by my ratio of circumference and diameter) =.7183315+. Now let the three sides of the equilateral triangle F=3, then each side of F=1,—the perpendicular or diameter of F (a b)=$\sqrt{.75}$ and $\sqrt{.75}\times.5$ (half the side of F)=.4330127+ which is the area of F.

Now let the area of the circle G=.4330127+(=F) then the diameter of G (by my ratio)=√.551328+, the perpendicular or diameter of H (*c d*)=√1.240489+ and the side of H=√1.653985+ and √1.240489+ × half √1.653985+ gives the area of H=.7183315+ which is also the area of the circle E whose circumference is 3.

PLATE XI.



Also if F and H (Plate XI.) be hexagons,—then if the circumference or 6 sides of F shall equal the circumference of E, and the area of G shall equal the area of F, then the area of H will also equal the area of E, and the same is true of an octagon or polygon of any possible number of sides and of *every ratio* of circumference and diameter of a circle which can be used. The examples given will enable any one to prove the truth by figures for themselves. *The proposition is therefore demonstrated.*

It is a remarkable fact, that among all the modern attempts at analysis of the circumference of the circle, there is not one which gives to circumference a fixed and definite locality, and I have never seen the man among the mathematicians of the schools, who, on a circle being placed before him, could point to the position of circumference and say, it is *there*, *in that place*, simply because the imaginary line without breadth, having no existence, it can therefore have no fixed and definite locality, either in fact or in imagination;—being in itself a mechanical fallacy, it can never be mechanically applied, and for the present purpose is therefore useless. The laws of geometry are undoubtedly the laws of perfect mechanics, and any thing which can have no existence, such as an ideal line *without breadth*, is a *mechanical fallacy*,—and it is a *mechanical truth*, that nothing can have a definite and fixed locality, or occupy a definite and fixed position in space, without magnitude. The definition therefore "position without magnitude," whether applied to a line or a point, means nothing else, and cannot understandingly be made to mean any thing else, but the *place* of a magnitude without the developed magnitude itself. The existence of any shape signifies *limit*, and it is evident that a circle, in order to *be a circle in nature* must have *limit*, and a boundary definitely located, and mechanically defined, which boundary is its circumference; otherwise a circle cannot exist in nature, not even in imagination, if imagination be definite. It is evident then that the circumference of a circle having a fixed and definite locality, it is therefore a mechanical line which *limits* the extension of the circle, forming a perfect boundary, and separating it from all surrounding material, and therefore it *must have breadth*, even though such breadth may be less than any material portion of the circle developed; such, for example, are the lines which make the points of cohesion in the diamond, or in metals; and such also are the lines which separate the particles of water and enable them to flow by each other;—being positive lines they must necessarily have breadth, though such breadth is evidently less than any particle of the water, the diamond, or the metal which can be divided again without the parts so divided losing their character *as water*, as a diamond, or as metal,—in other words resolving themselves into their original elements (see remarks on magnitude and infinities preceding prop. III., Appendix).

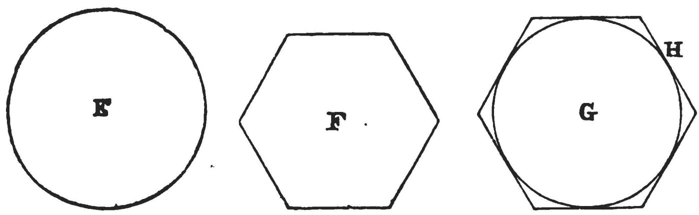
It would seem as if the definition of Euclid was sufficiently explicit to show us the position of circumference, and the term circumference would seem to mean neither more nor less than a line *circumscribing* and *inclosing* the figure. But the approximation of geometers is *not* a line *inclosing* or *containing* the figure, but a line *coinciding* with the greatest diameter of the figure, and is therefore *exactly equal* to the figure (proposition I., appendix), and if it have breadth, it is *a part of the area of the figure*, and therefore *incloses* or *contains less* *than the whole figure*, because no line can contain itself, and all that is within it: therefore—

PROPOSITION V.

The circumference of a circle by the measure of which the circle and the square are made equal, and by which the properties of straight lines and curved lines are made equal, is a line outside of the circle, wholly circumscribing it, and thoroughly inclosing the whole area of the circle, and hence, whether it shall have breadth or not, forms no part of the circle.

The demonstration of proposition v. is by the inverse of proposition IV. By proposition IV., it has been shown that if E (Plate XII.), be a circle of any certain circumference, and F be a polygon of any number of sides, and of equal circumference with E, G, a circle of equal area with F, and H a polygon circumscribing G, and of an equal number of sides with F, then the area of H equals the area of E. The inverse of the fourth proposition is therefore as follows. A polygon of any number of sides (H) is such, that the area of its inscribed circle (G) is equal to the area of another polygon (F), whose circumference equals the circumference of another circle, E, of equal area with H.

PLATE XII.



Now, therefore, E and H are equal in area, but E is formed of curved lines, and controlled by *their* properties, and H is formed of straight lines, and controlled by *their* properties; but it is evident, whatever may be the number of sides, that H is greater than G, and because G equals F in area, therefore H is also greater than F; and because H is greater than F, and F is a polygon of the same number of sides as H, therefore the circumference of H is greater than the circumference of F; and because the circumference of F equals the circumference of E, therefore the circumference of H is also greater than the circumference of E, and if brought into the form of a circle, will wholly circumscribe E. But the circumference of H will constantly approach more nearly to E by any increase of the number of sides of H, yet so long as H shall have the properties of straight lines, though the number of its sides were the greatest possible, its circumference shall always be greater than the circumference of E (proposition II., chapter i.), and if brought into the form of a circle, will wholly circumscribe E, though the difference may be such that it *cannot be less.* And in material things, any difference such that it cannot be less, is one ultimate particle of whatever material or thing is under consideration, which is also the essential difference between the properties of straight lines and curved lines. *The proposition is therefore demonstrated.*

By this demonstration, it is shown, that in order to give to the circumference of a circle the full expression of its value by the properties of straight lines, it is not to be measured as angular figures are, and after the manner of geometers, by a line *coinciding* with the greatest extent of diameter, but by a line *outside*, wholly inclosing the diameter, and the difference between the two is the compensation due to the circumference of the circle, to answer to the relative difference of property shown in proposition I., this chapter.*

* A very fine illustration from nature, of the difference between the line approximated by geometers, and the *true line of circumference* may be had, by placing a glass of water before us. If we suppose the tumbler to be a perfect cylinder, then, the surface of the water it contains will be the area of a circle. Now, the line of circumference approximated by geometers, is a line, lying wholly *inside* of every part of the tumbler, and *coinciding* with the *outer limit of the water.* The line which I say is the true circumference is the interior of the tumbler not coinciding with the water, but lying wholly outside of it, and inclosing the whole area of the water. It is evident that *every* part of the interior of the tumbler is farther from the centre of the circle than any part of the water, consequently the least possible line of the tumbler is greater than the greatest possible line coinciding with the water, because the one wholly *incloses* the other. It is evident, also, that the difference between the two lines is such, that

But this is not all the compensation which is due to the approximation of geometers, in order to make *that* equal to the true circumference; for in supposing the difference to be such "that it cannot be less," or only one particle, or least division of whatever material or thing is under consideration, I have only considered the difference between a line coinciding with, and a line inclosing the extreme diameter of a *perfect circle.* But geometers consider a *polygon* as equal to a circle, to the extent to which the sides of two polygons are brought to agree; and I have shown (proposition II., chapter i.), that it is *not equal.* I have also shown (proposition IV., chapter i.), that if the circumference of a polygon be brought into the form of a circle, both diameter and area are *increased*, and if diameter be considered as *fixed*, as it *is* by geometers, then, in order to give to circumference the full expression of its value in a circle, compensation must be made to the approximate ratio of geometers obtained from a polygon, of *four times the difference* of area between a polygon and a circle having the same circumference, and *both these differences*, which are errors

"*it cannot be less;*" because, in the nature of the water, it would by its own gravity adjust its particles to fill the whole circle, until there is not room in the same plane for one particle more in its least possible natural divisions. If therefore, the standard by which circumference is measured should be water-particles, then the difference between the least possible line of the tumbler, and the greatest possible line of the water, would be one particle of water in its least possible natural division.

of principle, are *additional* to all the errors arising from quantities neglected or lost in reducing their method to the calculation of figures. And both the errors of principle, and errors of quantities neglected or lost, are *increased* in value by increase of magnitude, as shown in proposition IV., chapter i.

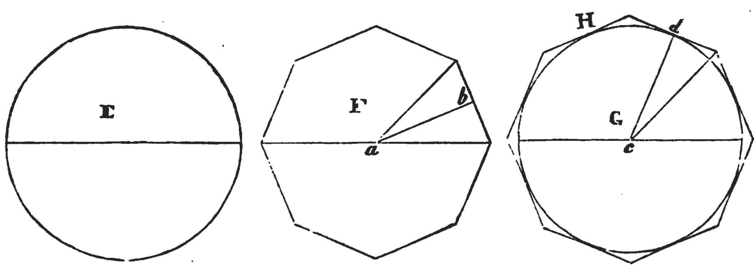
A portion of my papers were lately in the hands of a learned professor for examination, who returned for answer the very unphilosophical reason, that because "my ratio of circumference differed from the approximations of geometers in the sixth decimal place, he "therefore thought I must be wrong." In answer to such unworthy objections, and to aid in the development of the truth, I here add the sixth proposition, as follows:

PROPOSITION VI.

The circumference of a circle, such that its half being multiplied by radius, to which all other radii are equal, shall express the whole area of the circle, by the properties of straight lines, is greater in value in the sixth decimal place of figures than the same circumference in any polygon of 6144 *sides, and greater also than the approximation of geometers at the same decimal place in any line of figures.*

The foregoing proposition partakes of the general principle demonstrated in proposition II. and IV. Like them, therefore, it is true of *every ratio* of circumference and diameter which can be used, the approximate as well as the true ratio. For illustration, I will take the approximation of geometers as a ratio of circumference, viz., 3.1415926535 +, and let this be the circumference of a circle, and of a polygon.

PLATE XIII.



Let the diameter of the circle E = 1; then the circumference of E, by the above approximation, = 3.1415926535 +, and the area of E = .7853981633 +. Let F be a polygon of 6144 sides, in which the line *a*, *b*, is a perpendicular from the centre to either side of the polygon, and let the circumference or perimeter of F equal the circumference of E; *i. e.*, let it = 3.1415926535 +. Now, the perpendicular, *a*, *b*, mathematically determined after Playfair and Legendre's method, is found to equal .499999934636, and the area of F (half the circumference by *a*, *b*), = .78539806072+, which is seen to be less in the seventh decimal place, than the area of the circle E, having the same circumference. It has already been proved (proposition II. and IV.), that if G be a circle equal in area to F (Plate XIII.), and H a polygon of 6144 sides, circumscribing G, then the area of H equals the area of E. Now, therefore, let us see what is the circumference and diameter of H, which is equal in area to E. By the same approximate ratio of geometers (viz., the circumference of one diameter = 3.1415926535+), if the area of G equals the area of F, then the diameter of G = .999999934636 +, and the radius of G = 4.99999967318 +, and the radius of G is seen to be the perpendicular on either side, or least radius of H (c, d). If then, a, b (figure F), give for circumference 3.1415926535 +, then c, d, will give 3.1415928589+, which is the circumference of H, and half 3.1415928589+ × c, d, = .7853981633+ = the area of H, which is also the area of E. Now, let it be remembered that the circumference of F equals the circumference of E, and the *area* of H equals the *area* of E, but the circumference and diameter of H is greater than the circumference and diameter of F, by .0000002054+, which is in the seventh decimal place of circumference, and by .00000006536 +, which is in the eighth decimal place of diameter. And if the diameter of F be considered as *fixed*, and the whole expression of these values be given to circumference, in order to make F equal in area to E, then F should have a circumference = 3.141593+, which is seen to be greater than the approximation of geometers in the sixth decimal place of circumference, and greater also than the *so-called* perimeter of a circumscribed polygon.

The proposition is therefore demonstrated. And it is evident that if a circle, and a polygon of 6144 sides (the number to which Playfair carries his bisection) shall have the same circumference, the area of the circle is greater than the area of the polygon in the sixth decimal place; and because the circumference of one diameter must be four times the area of the circle, therefore, by the transition of shape to a circle, the true value of circumference is greater in the sixth decimal place than *any approximation which can be obtained from a polygon of* 6144 *sides*, whether inscribed or circumscribed.

This part of the subject is entitled to a more enlarged treatment, but wishing to make my preliminary demonstrations as brief as is consistent with my purpose of carrying conviction to the mind of any candid and careful examiner, I omit it here, and direct the reader to propositions I. and III. (Appendix), for a more extended examination of a many-sided polygon, and the errors to which it is subject.

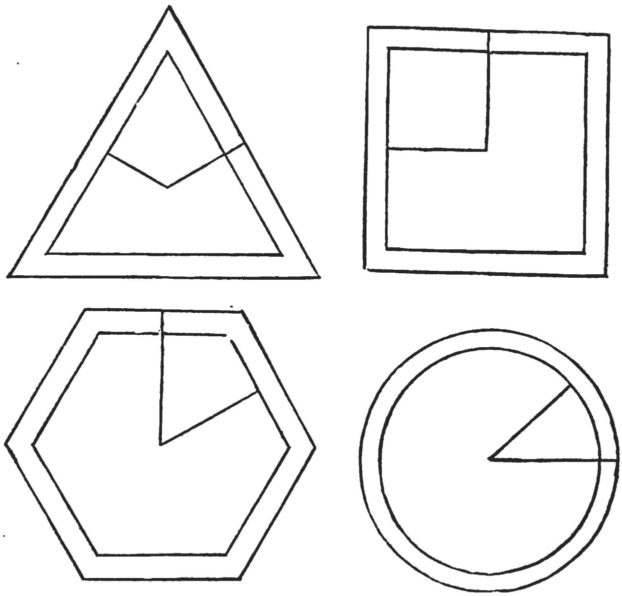
It will have been perceived ere this time, that I am fulfilling my first promise, that my reasoning would be "*wholly original*," and it will be perceived also, that my *mode of reasoning*, differs somewhat from the schools, in its elementary basis,—that, whereas, they reason wholly from the properties of straight lines (angles are a mere property of straight lines), I have taken up the properties of both straight lines and curved lines, and have introduced *shapes* as a new element of mathematical reasoning. I am authorised to do this by truth and nature, for it will readily be perceived, that the first step of nature in material creation, is, the production of *shapes*, and lines are nothing more (the imaginary lines of geometry) than the dimensions, boundaries, and divisions of shapes; therefore shapes are *primary things*, and hence a *true basis of mathematical reasoning.*

I have already shown (proposition III.), that the circle is *the primary* shape in *nature*, and hence the basis or beginning of all area, and that the square is only made the basis, by an arbitrary rule. I here add, as necessary to the course of development, another proposition, as follows:—

PROPOSITION VII.

Because the circle is the primary shape in nature, and hence the basis of area; and because the circle is measured by, and is equal to the square only in ratio of half its circumference by the radius, therefore, circumference and radius, and not the square of diameter, are the only natural and legitimate elements of area, by which all regular shapes are made equal to the square, and equal to the circle.

PLATE XIV.



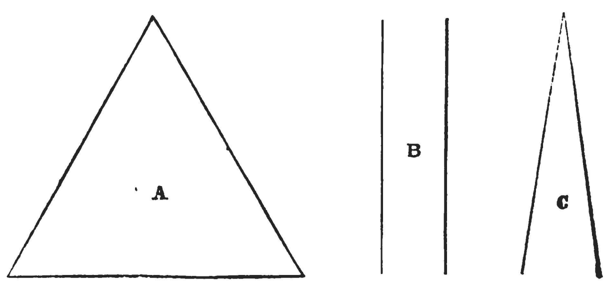
On examining the shapes one within another (Plate XIV.), it will be known to all mathematicians, without the necessity of a particular demonstration in each case, that the areas of any two or more equilateral triangles, squares, hexagons, or circles, are, to each other of the *same* shape, in ratio of area, as the squares of their diameters; but no triangle, square, hexagon, or circle, is, to either of the *other shapes*, in ratio of area as the squares of the diameters of each. Therefore, the negative part of the proposition, that the square of diameter is *not* the natural and legitimate element of area by which different shapes are made equal to one another, is proved. Referring to the figures again (Plate XIV.), it is seen, that any triangle, square, hexagon or circle, has an area equal to half its circumference by *the radius* (the *least* radius in all those shapes formed of straight lines), and consequently, they are all, to one another, in ratio of area, as *half their circumference by their radius.*

The proposition is therefore demonstrated; and circumference and radius are seen to be the only legitimate elements of area, by which all shapes, and all areas, are in like ratio to one another; and by which they are made equal to one another.

PROPOSITION VIII.

The equilateral triangle is the primary *of all shapes in nature formed of straight lines, and of equal sides and angles, and it has the least radius, the least area, and the greatest circumference of any possible shape of equal sides and angles.*

PLATE XV.



It will be seen that the triangle A (Plate XV.) has three equal sides, formed of three equal lines. Now if we suppose a shape formed of only two lines, as B or C, if such shape shall have breadth or magnitude, then it must have more than two sides, and if more than two sides, then it is formed of more than two lines, and if it has three sides, then it is a triangle. But if such supposed shape formed of only two lines has neither breadth nor magnitude, then it is not a shape, and is without existence. It is evident therefore that no angular shape *can have* less than *three* sides, and hence the triangle is the first production of nature, among shapes formed of straight lines, and, therefore, the equilateral triangle is the primary of *all shapes* formed of straight lines and *equal sides* and angles.

It is known that if the circumference of the triangle A be bisected at each side so as to form a hexagon in shape, the area of the hexagon will be greater than the area of the triangle. In other words, if the circumference of the triangle be brought into *any other shape*, such as a square or a hexagon, the area will be increased as the number of the sides is increased, and the radius is also increased in like proportion, circumference always remaining the same. It is therefore known that because the triangle has the least number of sides of any possible shape, it has also the least radius and the least area of any possible shape formed of straight lines of equal sides and angles and of the *same circumference.* And *because* the equilateral triangle has the least radius and the least area of any possible shape formed of straight lines, of equal sides and angles, and of the same circumference; therefore by reciprocity it is known, also, to have the *greatest circumference* of any possible shape of equal sides and angles and of the same area.

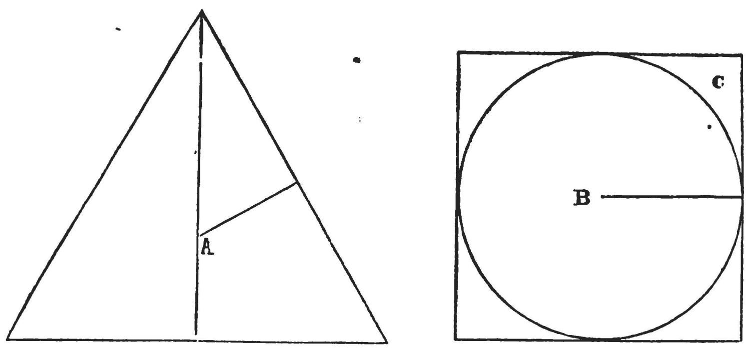
The proposition is therefore demonstrated, and it is thereby manifest, that because the circle and the equilateral triangle are the primary of all shapes in nature, one formed of curved lines and the other of straight lines, that *therefore*, the *primary* difference between straight lines and curved lines, and hence their *equality one to the other*, is to be found in the relations between the *circle* and the *equilateral triangle.*

PROPOSITION IX.

The circle and the equilateral triangle are opposite *to one another in all the elements of their construction, and hence the fractional diameter of one circle, which is equal to the diameter of one square, is in the opposite duplicate ratio to the diameter of an equilateral triangle whose* area is one.

By diameter of the triangle, the perpendicular is here meant, as explained in the introduction to chapter i., or a line passing through the center of the triangle, and perpendicular to either side.

PLATE XVI.



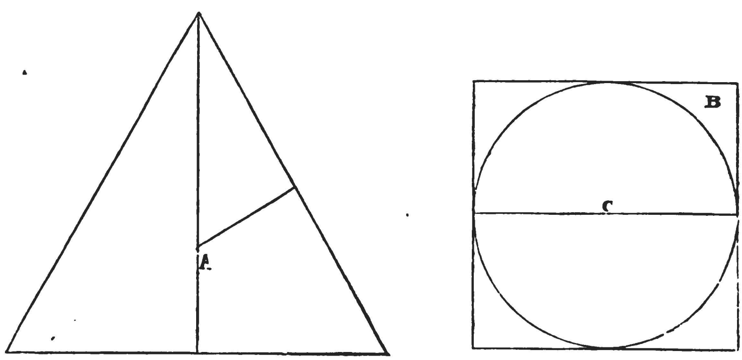
Let it be supposed that the areas of the equilateral triangle A and the square C each equals *one.*

It has been shown (proposition VIII.), that the triangle has the least number of sides of any possible shape in nature formed of straight lines; and the circle is the ultimatum of nature in the extension of the number of sides. In this particular therefore they are opposite to one another in the elements of their construction. By proposition VII. it is shown that circumference and radius are the only natural and legitimate elements of area by which different shapes may be measured alike, and are made equal to one another. By proposition VIII. it is shown, that the triangle has the *least* radius of any shape formed of straight lines of equal sides and of the same circumference, and by prop. II. and IV. chap. i. it is seen, that the circle has the *greatest* radius of any possible shape of the same circumference. By the same propositions the triangle is shown to have the *greatest* circumference and the *least* area of any shape formed of straight lines and equal sides, and the circle is shown to have the *least* circumference and the *greatest* area of any shape. By a well known law of numbers and geometry by which the greatest product which any number or any line can give, is, to multiply half by half, it will be seen that if we take the aggregate of circumference and radius in each shape, it is most *equally* divided in the circle, and the most *unequally* divided in the triangle, of any possible shape. In *every* case, that which is *greatest* in the triangle is *least* in the circle, and that which is *least* in the triangle is *greatest* in the circle, and in every particular the two shapes are at the extreme and *opposite boundaries of nature*, being the *greatest* and the *least* that *is possible.* They are therefore opposite to one another in all the elements of their construction. Therefore the square being made the artificial basis of area (prop. vii.), if the diameter of the circle B (Plate XVI.) shall equal the diameter of the square C, then, in the fractional relations of B to C such diameter shall be in the opposite duplicate ratio to the diameter of A correspondingly situated. The diameter of A correspondingly situated with the diameter of B to C, it will be seen, is a line drawn across the center of A perpendicular to either side; therefore the diameter of B in its fractional relation to C is the opposite duplicate ratio to the *perpendicular* or diameter of A. And no other result is possible in the nature of things (see prop. VII. Appendix and remarks following). *The proposition is therefore demonstrated.*

PROPOSITION X.

The fractional diameter of one circle *which is equal to the diameter of* one square *being in the opposite ratio to the diameter of the equilateral triangle whose area is one, equals* 81.

PLATE XVII.



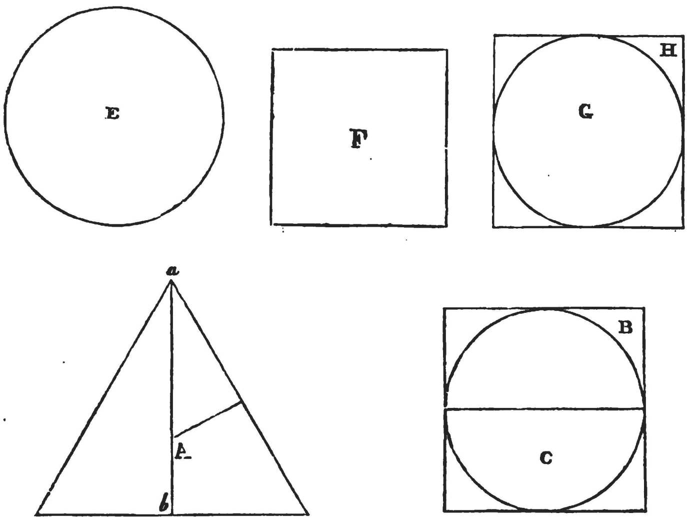
Let the area of the equilateral triangle A (Plate XVII.) =1, and let the area of the square B also equal *one*, then the diameter of the circle C, which is equal to the diameter of the square B, also equals *one.* And it has been demonstrated that in their fractional relations to the square, the diameters of A and C are in opposite ratio to one another. (By the diameter in the triangle it is known that the perpendicular is here meant, as in prop. IX.) Now if the area of the equilateral triangle A shall equal one, then the diameter of A (*a b*), is found to be equal to the square root of three twice extracted, or $\sqrt{\sqrt{.3}}$. Hence the fractional diameter of C, being in the opposite duplicate ratio (which is the square of diameter) shall equal 3 twice squared or $3^2 \times 3^2$ and $3\times3=9\times9=81$. *The proposition is therefore demonstrated.*

Under the head of "The opposite duplicate ratio of the equilateral triangle and the circle," embracing prop. VI. and VII. (Appendix), will be found a more particular examination of the points contained in the above demonstration, to which the reader is referred for his more perfect satisfaction.

PROPOSITION XI.

The fractional area of one square which is equal to the area of one circle, equals 6561; *and the area of the circle* inscribed *in one square equals* 5153.

PLATE XVIII.



It has been proved (prop. x.) that the fractional diameter of the circle C, which is equal to the diameter of one square (B) whose area is one, being in the opposite ratio to *a b* (figure A), equals 81; hence the area of B=81×81=6561; therefore B equals one, of 6561 equal fractional parts. Now let B equal H in area. It has been proved (prop. II.) that H=E in area, and if H=1 then E=1, and if H=6561, then E=6561. It has also been proved (prop. II.) that if the circumference of F equals the circumference of E, then F and G are also equal in area. And because one circle which is equal to one square (the area of the square being one) is in 6561 equal fractional parts, therefore *any* circle which is equal to *any square* (the diameter of the circle being a whole number) shall be in some definite and certain number of $\frac{\ }{6561}$ parts. Hence the areas of the circles C and G (their diameters being each 81) are some definite and certain number of $\frac{\ }{6561}$ parts of B and H. It is proved by the approximations of geometry obtained by the properties of straight lines, that C and G are each greater (much greater) than $\frac{5152}{6561}$ parts of B and H, and less (much less) than $\frac{5154}{6561}$, therefore (*Reductio ad absurdum*) they *shall* be each $\frac{5153}{6561}$ because they *can be nothing else*, there being no other $\frac{\ }{6561}$ part between 5152 and 5154.

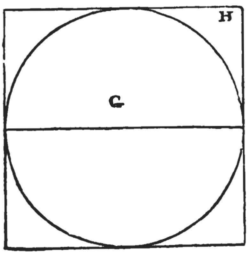
The proposition is therefore demonstrated; and the fractional area of one square which is equal to one circle (the area of each being one) is 6561, and the fractional area of one circle *inscribed* in such square is 5153.

It will now be seen that having determined two parts of each C and G, *i. e.* having determined the fractional area and diameter of each, if we divide the area by one-fourth the diameter, it will give us the circumference of each. And because the diameter of each C and G=81, and the area of each =5153, therefore 5153÷20.25 (¼ the diameter) =254+ with a remainder forever. It is evident, therefore, as has always been the case with others, that I have not yet reached a circumference which may be expressed in a whole number, and in decimal figures, without a remainder. I therefore add another and *final proposition*, as follows.

PROPOSITION XII.

The true ratio of circumference to diameter of all circles, is four times the area *of one circle inscribed in one square for the ratio of circumference, to the area of the circumscribed square for the ratio of diameter. And hence the true and* primary *ratio of circumference to diameter of all circles is* 20612 *parts of circumference to* 6561 *parts of diameter.*

PLATE XIX.



It will be known that if the diameter of the circle G inscribed in H (Plate XIX.) = 1, then the area of H also =1. It will be known also, that the area of G equals half the circumference, multiplied by half the diameter, and $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; hence the diameter of G being *one*, then the area of G equals $\frac{1}{4}$ its circumference, and vice versa, the circumference of G equals four times its area. And the diameter of G being one, it therefore equals the area of H, because the area of H = 1. Therefore, the first part of the proposition is demonstrated, and four times the area of any inscribed circle for a ratio of circumference, to the area of the circumscribed square for a ratio of diameter, is seen to be a true ratio of circumference to diameter of *all circles*.

It has been proved (proposition XI.), that by the primary relations existing between straight lines and curved lines, as developed by the opposite ratio of the equilateral triangle and the circle, the fractional area of H = 6561, and the area of G = 5153; *therefore*, the true and *primary* ratio of circumference to diameter of all circles = 4 G for the ratio of circumference to the area of H for the ratio of diameter; and since G = 5153, and H = 6561; *therefore*, the *true* and *primary* ratio of circumference to diameter of all circles = 5153 × 4 = 20612 parts of circumference to 6561 parts of diameter. *The proposition is therefore demonstrated*, AND THE QUADRATURE OF THE CIRCLE IS DEMONSTRATED!!

NOTE TO CHAPTER II.

IT will be seen by any one who has carefully examined the propositions in chapters i. and ii., that the whole demonstration of the quadrature rests on the last four propositions of chapter ii. All the others are only preliminary, to show the errors of the received mode of demonstration, and to open the way to this. The remark following the eighth proposition, chapter ii., is, I think, a conclusion fully warranted, viz., "that because the equi-"lateral triangle and the circle are the primary shapes "in nature (propositions III. and VIII.), one formed of "curved lines, and the other of straight lines, therefore, "the primary difference between straight lines and "curved lines, and hence their equality one to the other, "is to be found in the relations between the circle and "the equilateral triangle."

This most natural and conclusive inference, drawn from proposition VIII., and those preceding it, leads our reason at once, to examine the relative properties of the equilateral triangle and the circle, and this examination again leads us at once to the ninth proposition, viz., "that the equilateral triangle and the circle are *opposite* "to one another in all the elements of their construction, "which are concerned in plane extension, and hence the "square of diameter being made the artificial basis of "area (proposition III. chapter ii.), therefore the equilat-"eral triangle and the circle are opposite to one another "in ratio of the squares of their diameters."

This, then, is the only question of doubt to settle. If the equilateral triangle and the circle are opposite to one another in all the elements of area which enter into their construction, then are they also opposite to one another in ratio of the squares of their diameters.

But if they are not opposite to each other in ratio of the squares of their diameters, then are they not opposite to one another in the elements of their construction. This question, the examination of proposition IX. settles conclusively. And that proposition being proved, all the serial and algebraic formula in the world, or even geometrical demonstration, if it be subject to any error whatever, cannot overthrow the ratio of circumference and diameter which I have established! So long as it remains a truth, that the equilateral triangle and the circle are opposite to one another in the elements of their construction, that ratio of circumference and diameter will stand forever against all argument and all demonstration by the properties of straight lines which can be brought to disprove it, and time will show all the efforts of geometers to disprove this, to be just as idle as all their efforts to prove the value of the circumference of a circle by the properties of straight lines have been. These (the opposite elements, and opposite ratio) are *the particular facts* which govern the circumference of the circle in its relation to the properties of straight lines, according to the idea given out in proposition II., and which make the circumference of the circle to be a *particular fact also*, and *not* a general principle, though true to every thing that is a general principle; because, it is itself the basis of all *general principles* in relation to magnitude or area.

Having therefore arrived at that point where, by the natural conclusion of the course of reasoning adopted, the quadrature of the circle is demonstrated, I may now be allowed some freedom of remark, in respect to the credit due to those opinions, which for the last half century have condemned the quadrature as a useless question, which it was impossible to solve.

In respect to the utility and value of the quadrature, I think it will be sufficient to say, what no one will dispute,—that from the earliest time in the history of geometry, up to the close of the last century, it was considered by every mathematician, geometer, or astronomer, of eminence, to whom the world owes all that is known on these subjects, that the quadrature of the circle *was*, and *is*, an elementary truth, *necessary* to be known for the perfection of mathematical and astronomical science. In the course of the preceding demonstrations, I have shown clearly, I think, that all geometrical truth whatsoever in nature, rests on two simple things, viz., the properties of straight lines, and the properties of curved lines, and that the relations of these to one another are controlled by the circle and the equilateral triangle as *primary shapes*;—the circle, and consequently curved lines, being *primary of all others*; thus sustaining, and being sustained by, the opinions of all ancient geometers, that the quadrature of the circle lies at the foundation of all geometry, and hence it is an elementary truth, necessary to be known for the perfection of mathematical science. Under these circumstances, therefore, when I hear a learned professor of an exact science, stigmatize the quadrature as "a useless question," and one in which an approximation is "near enough," I am led to conclude, either that he does not understand the nature of the subject, or that his declaration is made to console a wounded pride, in not being able to reach it himself.

Foremost among those who have thrown discredit on the pursuit of the quadrature, has been Legendre, the eminent French geometer; and I confess to an abundant surprise at finding, that the professors of our own day and in our own country particularly, have received what Legendre and a few others have said, as *established facts*, and have adopted their opinions without investigation. A distinguished professor of one of our own distinguished colleges, to whom I sent some of my original papers, once wrote me in reply; and referring to Legendre's note on the subject of the quadrature in his elements of geometry, added, "that after seeing what that great geometer "had said, he presumed I would think no more of the "subject."

In the note in question, Legendre, after having finished a bisection to 8000 sides of a polygon, asserts that he has determined the quadrature as accurately as the root of any imperfect square can be determined to the same number of figures; and after concluding that the perfect quadrature is impossible to be found, he adds, in substance, "that no one having the least pretension to geo-"metrical science will ever make the attempt"!!!

What consideration is due to these declarations of Legendre, may easily be seen. He did not even attempt to measure the exact circumference of a circle; and he did not in *the least* consider the properties of curved lines, notwithstanding that it had long been known that curved lines do possess properties essentially different from straight lines; yet Legendre, by his method, tacitly admitted that he knew nothing about them. He simply bisected the perimeters of two polygons on a given radius, until he brought their sides to agree with each other to a certain extent, and these polygons he then considers as equal to a circle, to the extent to which their sides agree!! But I have demonstrated (and any one may know the fact, almost without a demonstration), that a circle is greater than a polygon having the same circumference at any possible number of sides (proposition II., chapter i.); especially is it greater than a polygon of only a few thousand sides. The conclusion of Legendre is therefore manifestly erroneous. He measured nothing but polygons, and judging from what he has said, he neither understood nor regarded the effect of transition of shape to a circle. (See remarks following proposition IV., chapter i.)

Again, it will be obvious to all, that by Legendre's method, his errors from quantities neglected or lost in the course of his calculation and reduction of his result to numbers, are errors of *each side* of the polygon, and consequently while in that result, he can have but one error in his diameter, he has 8000 errors in his circumference, and he could not tell whether the sum of these errors was plus or minus!! And again, these errors are not errors in the circumference of a circle, but *in the perimeter of his polygon*, which is of less value than a circle having the same circumference (propositions II. and IV., chapter i.), and hence they stand additional to the errors arising from the essential difference in the properties of straight lines and curved lines, which he has wholly neglected. Yet in the face of these facts, which he must have known if he understood his own work, Legendre has seen fit to declare that he has determined the circumference of a circle "as accurately as the root of any imperfect square can be determined." And who, I ask, will, after such a declaration, feel his pointed rebuke, or trust what he may say on this subject, without first satisfying himself of its truth? For my own part, I cannot but believe that Legendre's reputation as an eminent geometer, great though it deservedly is, would have been more enduring if he had left the note in question out of his work entirely.

Playfair of Edinburgh was nearly contemporary with Legendre, and his and Legendre's Elements of Geometry, or rather their editions of Euclid, have for a long time been standard works in the English language on these subjects. Playfair admits that his and Legendre's method of determining the value of circumference (which was also the method of Archimedes 2000 years ago) is *defective*, but modestly says, that geometers "know no better method." Still, however, Playfair is supposed to have sided with Legendre in thinking that the exact solution of the problem was unattainable.

Legendre and his coadjutors were members of the Academy of Science in Paris, and Playfair and his coadjutors were members of the Royal Society of London.

About the era of the publication of Legendre's work, the Academy of Science in Paris, instigated perhaps by Legendre himself, passed a resolution, that, in order to discourage such futile attempts, the Academy would not thereafter receive or consider any paper purporting to be on the subject of the quadrature! And within a few years thereafter the Royal Society of London passed a similar resolution!! And under the influence of these tyrannical proceedings it soon became disreputable in learned circles for any one to talk of finding the quadrature. But these resolutions of Academies and Royal Societies needed some support, and Montucla generously comes to their aid in his *History of the Quadrature*, which, so far as I can judge from such portions of it as have been brought to my notice, appears to have been written for the purpose of pandering to the prejudice of the French school, rather than to do justice to the many ingenious but ineffectual attempts which have been made to solve the problem. He remarks with an air of complacency, which he seems to think is a conclusive argument, "that he never knew a man who thought that "he had discovered the quadrature who would ever be "convinced of his error, however clear the argument "might be against him." In this remark he seems to have forgotten the possibility that he and his associates were afflicted with the same human infirmity,—an unwillingness to be found in error; but if we examine into the history of the progress of science we shall find, that the great stronghold of this mental disorder has always been found within the walls of the academy. If this remark needed any evidence to sustain it, I might instance the almost uniform and steady resistance with which the schoolmen have received the first introduction of every great discovery in modern times.

Here then is a brief history, and the whole merit, of that popular prejudice which for the last fifty years has condemned the quadrature of the circle as a useless question which it was impossible to solve, and the only very remarkable thing about it is, that the really intelligent portion of the mathematical world should so long have bowed in meek submission to the tyranny.

The only demonstration ever made or ever pretended to be made, by any body, of the impossibility of finding the quadrature, is only a demonstration that the thing is impossible to be done by means of straight lines alone, and geometers have never used any thing else but straight lines. I have never disputed nor even doubted the truth of this fact,—far otherwise, and so plain and simple is it, that it hardly needs a demonstration, but may be easily understood by almost any one of only moderate mathematical perception. And any one who thinks that there is necessarily any disagreement between what I profess to have proved and what has been proved by the properties of straight lines, or any thing contained in Euclid, will find himself, upon a full examination, to be greatly in error.

I also have demonstrated the same thing, and not only so, but my demonstrations show clearly, I think, that no straight line, nor any number of straight lines *in a circumference*, can in any possible shape or position, be equal in relative value to the same length of curved line. Hence, instead of pursuing the quadrature as all others have done, by reasoning wholly on the value of straight lines, I have first turned my attention to discover by means of shapes, some of the relative properties of curved lines to straight lines, and then I have made use of these properties to determine their relative value,—a mode of proceeding, I believe, never before adopted in any attempt to solve the problem; and few, I think, will disagree with me in believing that this is the only practicable mode. Some may, and no doubt will, find fault with it. I expect this, from the influence of that prejudice, of which I have just given the brief history, for it is hard, for scientific men above all others, to yield a fostered prejudice. But there are many men, even among the most learned of mathematicians, whose power of reasoning is limited to the rules which they have studied, and any thing which comes not within *these rules* is, to their minds, not mathematical. Such men are often heard to say, that "the science of mathematics is "perfect, and that there can be no change or improvement "in it; because," say they, "mathematics are true and the "truth cannot change." But such men, I think, confound the *written science* with the truth,—they substitute a mere method by which truth is determined, for the truth itself, when in fact there is only an affinity between them.

Nature never follows the rules of mathematicians, though she is in fact a much better mathematician than any of them; she arrives at the same results however by methods of her own, and the nearer our methods approach to hers, the more simple and perfect they are.

The whole written science of mathematics is, to my mind, nothing more than a process of inductive reasoning, wherein by laying hold of certain primary and self-evident truths, we thence deduce other truths, and from these again yet others, constantly widening the basis of our reasoning and always ascending from the lower to the higher. In geometry the primary truths adopted, are, simply, a straight line and a given angle, which form the starting points, and from these, all subsequent deductions are made. The whole course of geometrical reasoning therefore which is known to geometers, is based wholly on the properties of straight lines, and all circular or spherical measures as yet known or used, are, only approximations of straight lines, which in circular or spherical magnitudes are both erroneous in principle and subject to error in reducing their value to numbers. Other systems of geometry might have been formed which would have been equally accurate, yet based on other shapes, and other truths, but whatever might have been adopted as a basis of reasoning, numbers in some one of their infinite capacities of notation and combination, and not straight lines alone, would lie at the bottom of all truth. It is not necessary, therefore, in order that our reasoning should be strictly mathematical, that we should follow the rules laid down in the written science for reasoning on the properties of straight lines. It is only necessary that we should abide by the *same principles* which are laid down for that purpose, viz. *to seek first from nature* some primary and self-evident truth from which to deduce other truths; and then it is only necessary, that the truths deduced, should be *necessary results* of the primary truth which forms the basis of our reasoning, and if we abide by these principles then our reasoning is *strictly mathematical.*

Hence whether we reason from the properties of numbers or magnitudes, shapes or areas, lines or angles, or from all of them together, if our deductions are necessary results of premises founded on the truth of nature, then such deductions are *mathematically true.* Hence I am justified in speaking of a line and an area as equal to one another (prop. XII.). Hence also I am justified in introducing shapes as an element of mathematical reasoning independently of lines, because shapes are *primary* *things* to which lines are only secondary, and of which they constitute the divisions, dimensions, and boundaries, as stated in the remarks preceding prop. VII.; but without shapes lines could have no existence and would become wholly inoperative.

Every step which I have taken and every conclusion at which I have arrived in the course of these demonstrations, will be seen to be founded in, and a necessary result of, some primary and self-evident or demonstrable truth, and therefore the result *cannot fail to be true.*

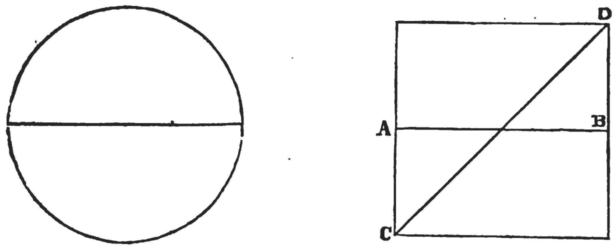
In mathematics there is no room for opinion; what a man demonstrates by unerring principles he either *knows* to be true, or he does not understand his own work; and believing that I have understood mine, in submitting these demonstrations to public examination and criticism, I do not ask whether the result is true or not, but I assert that it *is true*, and I hold myself ready to sustain it. Fortunately, however, for my time and industry, the truth, if such it be, being once known, will be able to sustain *itself* without *my* aid.

It is proper in connection with this note to remark, that the method of demonstration which I have here used, is not the only method by which I have arrived at the same result, nor is it the method by which the ratio of 20612 to 6561 was first discovered. That ratio was first discovered by reasoning wholly on the properties of numbers, independently of lines or shapes, and assuming as a primary truth, *first*, that in the material creation, numbers and things (magnitudes or shapes) are identical and inseparable. *Secondly*, (as has here been demonstrated) that the circle is the beginning of all magnitudes or areas (prop. III. chap. ii.), and hence the *beginning of numbers also;* and *thirdly*, that consequent upon these primary truths, numbers themselves are constructed upon the circle, and hence by reciprocity, the *truth of the circle is the basis of numbers.* Reasoning from these premises, I arrived at the conclusion that the circle in its relation to the square (the area of each being one) is composed of 6561 parts, and taking advantage of approximation I inferred, "reductio ad absurdum," that the ratio of circumference to diameter is 20612 to 6561.

In conformity with the premises above assumed it will be seen, that the fraction 6561 divided into its own root (81) produces a repetition of the digits to infinity, the number *eight* being always missing in consequence of our use of decimal numbers in dividing. Also if $\sqrt{6561}$ (81) be divided into *one*, the product is the same—a repetition of the digits to infinity, thus conclusively showing that this fraction is a basis of the digits used in the construction of decimal numbers.

The second method by which I arrived at the same result was by dissecting a circle and a square into the *original whole* and *separate* parts necessary to form a circle and a square in *shape* and *area*, as follows:

PLATE XX.

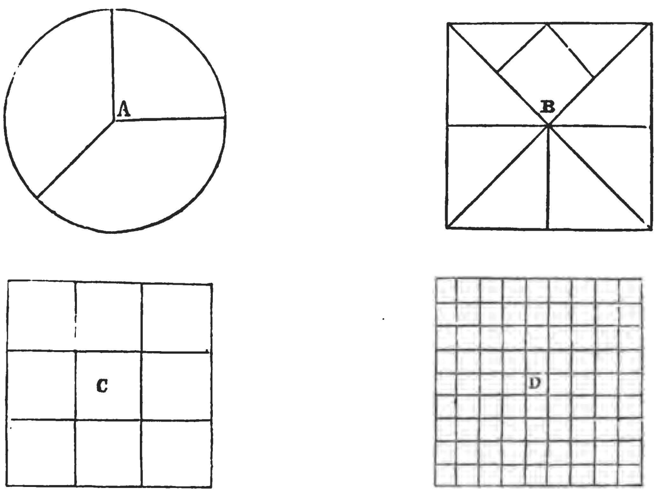


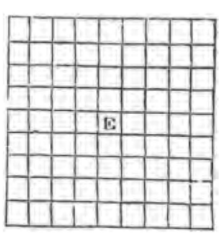
Let us suppose that the circle and the square (Plate XX.) are equal to one another in area, that is each =1. It will be seen that the parts necessary to form a circle in shape and area are *three*, viz., it has one continued line for circumference, and therefore its circumference is one, —it has but one diameter (all other diameters in it being equal) and therefore its diameter is *one* (of a circle)—the square of its diameter is also *one* (of a circle) making in all *three original whole* and *separate* parts; and these are seen to be *all* the parts that are necessary to form a circle in *shape* and *area*, and without these parts a circle cannot be formed.

The square is seen to have four distinct and separate lines for its circumference, and therefore its circumference is *four*. It has two diameters, viz., A B and C D (the greatest and the least) and therefore its diameters are *two*. The squares of its diameters are three (of a square) viz. A B^2=1 and C D^2=2, making in all *nine original whole* and *separate* parts, and these are seen to be *all* the parts necessary to form a square in *shape* and *area*, and without these parts and in these exact proportions a square cannot be formed.

In the two shapes, however, these parts are, in their relation to one another, wholly incongruous in quantity. Their equality to one another is therefore to be found by the following method.

PLATE XXI





Let us again suppose that the circle A, and the square B (Plate XXI.), each equal one in area. It has been seen that the circle is composed of three original, whole and separate parts, and the square of nine parts. Now, let us suppose that there are three magnitudes contained in A, of any shape, and of incongruous quantities, and that there are nine other magnitudes contained in B, of any other shape, but individually and relatively to one another, and to the magnitudes contained in A, wholly incongruous in quantity, but the sum of the three magnitudes, and the sum of the nine magnitudes are each equal to one another; and for illustration, let these magnitudes be represented by the areas contained in the lines drawn on A and B. It is now required to find the perfect equality of these incongruous magnitudes in the parts of the square. Therefore, because $A = 3$; therefore $A^2 = 9$; and we have the figure C equal to the circle A, equal to the square B, and equal to the nine magnitudes contained in B, but the individual magnitudes contained in C and B are unequal to one another; therefore, because $B = 9$, then $B^2 = 81$; and we have the figure D equal to the circle A, equal to the square B, and equal to the whole of C, but the separate magnitudes contained in C and D are unequal to one another; therefore, because $C = 9$, then $C^2 = 81$; and we have the figure E, equal to the circle A, equal to the square B, and equal to D, and all the separate magnitudes contained in D and E are equal to one another *in the parts of the square.*

Now, therefore, because 81 is the smallest number by which the unequal magnitudes of incongruous quantity contained in A and B can be made equal to one another *in the parts of the square*; therefore, 81 is a diameter by which the fractional area of B equals the area of A, and hence the fractional area of A, in its relation to B, = E^2, and the fractional area of B, in its relation to A, = D^2, and because E and D each = 81, therefore the fractional area of A and B each = 81 × 81 = 6561. Therefore it is demonstrated that the fractional area by which one circle is equal to one square (the whole area of each being *one*), is in 6561 equal fractional parts.

Now, therefore, since B = 6561, by proposition XI., chapter ii., it is proved that the area of a circle inscribed in B = 5153, and by proposition XII., it is proved that the true ratio of circumference to diameter of all circles is four times the area of the circle inscribed in B for a ratio of circumference to the area of B for a ratio of diameter; therefore, the true ratio of circumference to diameter of all circles is 20612 parts of circumference to 6561 parts of diameter. Q. E. D.

In proposition III. (chapter ii.), I have demonstrated that the circle being the primary shape in nature, is therefore the natural basis or beginning of all area, and the square is the artificial basis created by science.

From this conjunction of facts, it will be seen, that because one shape (the circle) forms the basis of nature, and the other shape (the square) forms the basis of art, therefore, in the elements of their construction, or in the component parts necessary to form a circle and a square in shape and area (the area of each being one), the original whole and separate parts of the two shapes *shall also be equal*, or the parts of one shape shall be the root of the parts of the other shape. Therefore, it is seen in the foregoing demonstration, that because the circle is *primary*, therefore the parts necessary to form a circle in shape and area are the root of the parts necessary to form a square in shape and area ($\sqrt{9} = 3$), and although these parts, in their relation to one another, are originally and individually wholly incongruous in quantity, yet because one is the root of the other, their perfect equality is readily found, as in the demonstration.

It will be perceived by any one who has read with any care, that this method of showing the relations between the circle and the square to consist of 6561 fractional parts, has no similarity to, or dependence on the opposite duplicate ratio; on the contrary, it is entirely dissimilar, and wholly independent of it in its principles and operation. I consider it perfect in itself, yet I do not advance it here as any part of my argument, but rest the decision of the truth entirely on the opposite duplicate ratio of the equilateral triangle, and the circle as explained and demonstrated in the twelve propositions of chapter ii.

I have introduced these two last methods of finding the quadrature only into this note, and have forborne to treat them at length on their merits, lest I should make my work too large. I would remark, however, that I hold a mass of papers and correspondence in respect to this last method (which was the second in the course of discovery), and which I may make use of at some other time. I consider *one method sufficient*, and I have preferred the method of demonstrating by the opposite duplicate ratio of the equilateral and the circle, which was the third in the course of discovery, as being to my present views the most full and complete. But it is due to the merits of the second method, by dissecting of the parts, to say, that it was the examination of the subject by this method which revealed to me many of the properties of curved lines, and led to the discovery of the third from the opposite duplicate ratio.

The methods which I have used may doubtless be somewhat improved upon, and there are various other methods which may be used for the purpose with equal effect, but if we have found the truth, it is useless at present to descant upon these.

CHAPTER III.

PRACTICAL QUESTIONS.

It will be understood, from what has been said in the preceding chapter and note, that I claim for the ratio of circumference and diameter which I have established, that it is *the primary* ratio—*that* which is *one in nature*, and on which all other *true ratios* depend. I therefore denominate it, "*the primary circumference*," and a circle having its component parts, as a *primary circle.*

All circles are of themselves and in their own nature, primary circles, because all circles, without regard to magnitude, have the same relative and constituent parts, —that is to say, a small circle contains within itself the value of just as many angular spaces, and has the same ratio of circumference to diameter, which a larger circle has, the only difference being in the magnitude of those parts. But when we set aside any limited and definite quantity, calling it *one*, and set that up as a standard of measure by which to determine other quantities, then all other circles either greater or less than such standard, cease to be primary in their relations to that standard.

In introducing practical questions here, let it be understood, that it is not my object to write an elementary book of instruction, but only to test the applicability of *the primary ratio* of circumference and diameter to practical purposes, and at the same time to test the *truth* of that ratio by applying it to other *primary truths* in nature, the sole object and value of the discovery being its applicability to practical purposes.

In adapting the ratio of circumference and diameter of a circle to mechanism of any sort, it is evident from the almost inappreciable difference between my ratio and that of geometers, being scarcely equal to one hair's breadth in any arch ever constructed by men, that the only application of it which can be made to any natural truth as a test of its accuracy is, to the great astronomical circles, in whose vast magnitude alone, the value of the difference becomes important or even perceptible. I therefore proceed to make the mechanical application of my ratio of circumference to these great circles, upon the principles of mechanical motion, and in a manner which I think is peculiarly my own, but which I believe to be undeniably correct and a demonstration in itself. But in order to be understood, and to fix the attention of the examiner on the points necessary to be considered, I must first state a few preliminaries as follows.

First, Time (or perhaps I should be better understood by saying the standard by which time is measured) is nothing else but the relations existing between light and motion. Therefore time is *altogether relative*, and the motion of the revolving bodies by which time is measured, being measured by time, is *also relative.*

Secondly, Time and space are, for all purposes of calculation with respect to motion, *one and the same thing*, because the measure of time is the circumference of a circle, and its length or duration is the revolution of a circle. Therefore the circumference or area of *one* circle may be reduced to time, and the length of a day or a year may be considered and treated as circumference or area.

The area of one primary circle (5153) being reduced to time (corresponding with the revolution of one solar day) until it is without remainder, is 23h. 51′ 23″ 20‴.

One siderial day, or the revolution of the earth on her axis from opposite a fixed star to opposite that fixed star again, is 23h. 56′ 4″ 6‴.

The length of one solar day is exactly 24 hours, and for convenience I will call these a *circular day*, a *siderial day*, and a *solar day*, the solar day being greatest, and the *circular day* least of all the three.

Reduced to their lowest denominations, these three days stand as follows:

The length of 1 "Circular day" is 5153000‴.

The length of 1 Siderial day is 5169846‴.

The length of 1 Solar day is 5184000‴.

The difference between 1 Circular and 1 Solar day is 8′ 36″ 40‴.

The difference between 1 Circular and 1 Siderial day is 4′ 40″ 46‴.

The difference between 1 Siderial and 1 Solar day is 3′ 55″ 54‴.

The *excess* of difference between 1 Circular and 1 Siderial, and 1 Solar and 1 Siderial day, is 0′ 44″ 52‴.

Let it now be understood, that in computing the motion of a circle by time, we are to bring it at last to *solar time.*

Thirdly. All natural periods of time are, I think, (in accordance with the above table) greater than one primary circle, because all the heavenly bodies by whose motions time is measured, or whose motions are measured by time, are themselves also in motion. For example, the earth is *more than* 24 hours in revolving on her axis from the moon to the moon again, so as to bring the same meridian exactly opposite. But the earth revolves on her axis from the sun to the sun again in 24 hours exactly, and a lunar day is greater than a solar day, therefore the moon is in motion, and the earth performs on her axis more than a complete revolution in space in turning from opposite the moon's center to opposite that center again. But the earth revolves on her axis from opposite a fixed star to opposite that fixed star again in 23h. 56′ 4″ 6‴, and a solar day is greater than a siderial day, therefore either the earth or the sun is in motion in an orbit, and in either case the earth performs in space more than a complete revolution on her axis in turning from opposite the sun's center to opposite that center again. But the difference between a circular and a siderial day is greater than the difference between a siderial and a solar day, therefore *both* the *earth and the sun* are in motion in an orbit.*

The above are mechanical truths, easily proved, and if they be true, as I affirm, then the relative motion of the sun and of the fixed stars, so-called, may at length be found and demonstrated by the ratio of *one primary circumference!!*

Before proceeding to apply the ratio of circumference to the astronomical circles, it is necessary, first, to solve the problem of three gravitating bodies. I therefore submit the following proposition.

* The evidence of this fact does not depend on the truth of my ratio of circumference, and if the geometer's ratio *were true*, the difference would be greater still.

PROPOSITION I.

The respective and relative motion of three gravitating bodies revolving together and about each other, is as four to three, or one and one-third of one primary circumference.

I have always considered this proposition as self-evident on the face of it, and that no mathematician would deny it and hazard his reputation on sustaining the denial with proof. But as I shall perhaps be called on for proof, I add here, at some length, the solution of the problem, after my own method, as follows:

The problem of three gravitating bodies revolving together and about each other, is one, which, like the Quadrature, has hitherto baffled all attempts of mathematicians to solve it. But since this, like others of the kind, is of itself a problem, which is daily performed and consequently solved by the mechanical operations of nature, the failure of mathematicians to reach the solution proves nothing but the imperfection of the reasoning applied to it.

It is a principle I think clearly demonstrable, that whatever can be constructed by mechanics out of given magnitudes, can be *exactly* determined by numbers, and that which cannot be constructed by mechanics out of any given magnitudes, cannot be exactly determined by numbers, having the same relation as the magnitudes one to another. It is for this reason, and for this reason only, that we cannot *out of* the *same magnitudes* construct a square which is just twice as big as any other perfect square, neither can we find the perfect root of such a square by decimal numbers. If this reasoning be true, then, because the problem of three gravitating bodies is a mechanical operation daily performed in nature, it is hence a thing capable of being proved by numbers. The great difficulty of this problem has arisen, I think, from the impossibility of its full display by diagram, and the difficulty of embracing in any formula, all the conditions contained in its elements. The plan of exacting a display by diagram of all geometrical propositions is *safe*, and perhaps it is the only plan by which the yet untaught mind can be initiated into the truths of geometry, but is it always necessary in every original demonstration? Are there not other means equally true and *equally safe* in the hands of one accustomed to examination, and acquainted with the properties of numbers and of shapes? I think there are, and without taking the least unwarrantable latitude, or departing from the clearest perceptions of reason, I think this problem may be easily and accurately solved.

The thing required of every demonstration is, that it shall give a *sufficient reason* for the truth which it asserts. But in order that a reason may be *sufficient*, and the conclusion drawn from it *safe*, it is necessary, not only, that the relations of cause and effect shall be made clear to our perceptions, but also that the conclusion *when drawn*, shall abide the test of practical application. Any demonstration which does less than this cannot be relied on, and no demonstration ever made has ever done more than this.

We know very well that things are possible or impossible to be done, only in proportion as the means applied are adequate or inadequate to the purpose. We know also, that because different principles exist in the various forms of matter, therefore it is impossible to demonstrate every thing by the *same means* or *same principles*. It is a narrow-minded prejudice, therefore, which exacts that every demonstration shall be made by the prescribed rules of science, as if science already embraced every principle which exists in nature. Yet none are more frequently guilty of this narrow-mindedness than mathematicians, who often require that things shall be done by the means which the written science affords, well knowing at the same time that such means are *inadequate*. Such has always been the case in respect to the quadrature of the circle,—mathematicians have demanded that it should be demonstrated by the properties of straight lines, knowing at the same time, that straight lines are *inadequate;* therefore (*and therefore only*), the thing has been found impossible, and all other demonstrations are rejected, because they cannot be shown by straight lines. I do not consent to such unreasonableness of decision, but in every proposition where the *sufficient reason* is manifest, I hold the proposition to be demonstrated until it can be disproved.

In entering upon the solution of the problem of three gravitating bodies, we must first examine and see of what elements the problem is composed.

The elements which I shall consider in this case, will not be such as a mathematician of the schools would think it necessary to consider. They will be far more simple,—more conclusive (for such as the schools can furnish, have yet decided nothing), and I think, more comprehensible, yet equally true to nature (for I consult nature's laws only, and not the method or opinions of any other man), and equally accurate and precise with any which can be given by any other method.

And *first*, each revolving body is impressed by nature with certain laws making it susceptible of the operation of force, which being applied, impels motion. These laws may all be expressed under the general term *forces*, which, though various in their nature, possess an equalizing power, controlling each other in such a way, that neither can predominate beyond a certain limit; and, consequently, these bodies can never approach nearer to each other than a certain point, nor recede from each other beyond another certain point. Hence these forces are, at *some mean point*, made *perfectly equal*, and therefore they may be considered as but *one force*, and hence but *one element* in the problem.

Secondly. These revolving bodies have magnitude, shape, density, &c., which affect the operation of force in producing motion. These properties of revolving bodies have all the same inherent power of equalization as forces. For example, if density be greater in one than another, then magnitude will be relatively less, force will be less (the direct force), and the momentum from velocity greater, but the whole shall be equal. On the other hand, if magnitude be greater, and density less, then force will be greater, and velocity less, but the whole shall be equal. The second element of this problem may therefore be comprehended under the term *magnitude*, which shall include shape, density, and every other quality or condition which affects the operation of force in producing motion, and the whole constitute but one element in the problem, which I term *magnitude*, as referring to the bodies themselves rather than to any of their qualities, as density, gravity, or otherwise.

The *third* element in this problem is *distance*, by which I would be understood to mean the *chosen dis-*

tances from one another, at which these bodies perform their revolutions in space. It is well understood, that from the nature of the case, these revolving bodies must take up their mean distances from one another in exact proportion to their respective magnitudes and forces, and in proportion as these are greater or less, the distance from each other will be greater or less. Hence it is seen that the same inherent power of equalization exists in respect to distances as in respect to the forces and magnitudes, and whether their distances from each other be greater or less, equal or unequal, they still constitute but one element in the problem.

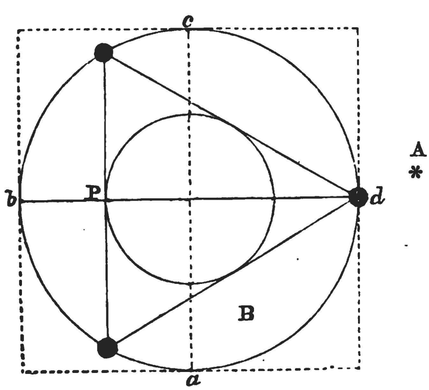
The *fourth* and *last element* in this problem is *motion*, or *velocity*, by which distances are to be performed or overcome by revolution. And here again it will be seen, that because the distances to be thus performed by revolution depend entirely on the *chosen distances* from one another, and these again depend on magnitude and force, therefore, the same equalizing power exists in regard to *motion* or *velocity*, as exists in regard to all the other elements, and therefore this also constitutes but one element in the problem, which I will term velocity, as including momentum, and every other quality, condition, or effect of motion.

These, *four in number*, are all the elements necessary for the *mechanical* performance of the problem, and consequently all that are necessary for its *determination* by *numbers*, and it has been seen, that such is the nature of the problem itself, and the power of these elements over one another, that every other quality or condition affecting either, is equalized by, and held in subservience to these, and these again are equalized by, and held in subservience to *one another*, and all controlled by magnitude, so that the whole constitute but *one problem* or *mechanical* operation, in which *four elements* are concerned.

The difficulty of reducing impalpable things to a palpable standard of measure is generally conceded, but in this case, I think the difficulty does not exist, and that these elements may all be as truly represented by numbers and magnitudes, as if they were palpable things in themselves, having the qualities of length, breadth, and thickness. For example, let a stone be a magnitude having shape, bulk, density, &c. Now, a *force* which can raise this stone one foot from the ground, and hold it suspended there, is, in its relation to the magnitude or stone, exactly equal to *one foot* of measure, and because the stone is held suspended, and does not descend again, nor rise higher, it is evident that the force and magnitude have *become equal at that point of elevation*, and therefore, *vice versa*, the magnitude or stone is, in its relation to the force, exactly equal to *one foot* of measure, and consequently distance and motion are each also seen to be equal to one foot; and the same principles of applicability to measure exist in *three* bodies suspended in space, and made to revolve about each other, by forces inherent in themselves. It matters not that other and disturbing forces exist outside or inside the space in which these bodies revolve, because, if another and disturbing force be considered, then it ceases to be a problem of *three* gravitating bodies; and also, because such disturbing forces, if they exist, operate proportionally on all *three* of the revolving bodies, and in the course of a revolution, and consequent change of *relative position*, these disturbances *must find* their perfect equality.

Now, let us suppose that we have here, three bodies revolving together in space by their own gravitating power, and let the magnitudes of these bodies be exactly equal to one another,—then their forces shall be equal, their distances equal, and their velocities equal, and it will be seen that they *cannot* revolve *about each other*, but must *follow each other* round a common center, and their relative motion in respect to any point in space (as the point or star A),

PLATE XXII.

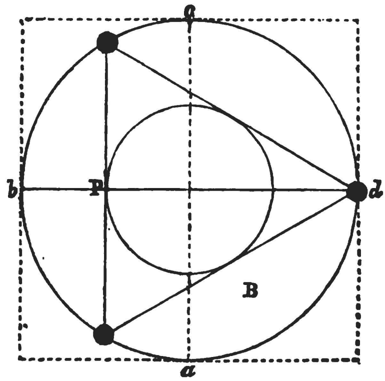


must be on the value of the circumference of the circle B, which passes through the center of each body, as in Plate XXII.

Now, let us suppose that each of the elements contained in the problem of *three* gravitating bodies, is an equal portion of the *area* of the circle which these bodies describe in a revolution; then the circle will be divided from the center into four equal parts, as at the points *a*, *b*, *c*, *d*, and let each part equal *one*. It will be seen, that in each relative change of position, each revolving body passes over an area equal to *one and one third*. In other words, their relative motion is as *four to three*. So also, if each element shall be an equal portion of the circumference of the circle B, or an equal portion of the square of the diameter of B, the same result is manifest, and the relative motion of each revolving body is, as *four to three* of such magnitude as is made the standard of measure. Again:

Secondly. Let the area of the circle *inscribed* in the equilateral triangle, whose sides make the distance between these revolving bodies, be *one*, as in the marginal Plate XXIII. It is seen that the circle B, whose circumference these bodies describe by their revolution, is four times greater than such *inscribed* circle. (See illustration, Plate XXXI., Appendix. Hence again, their relative change of position is seen to be as *four to three*, or

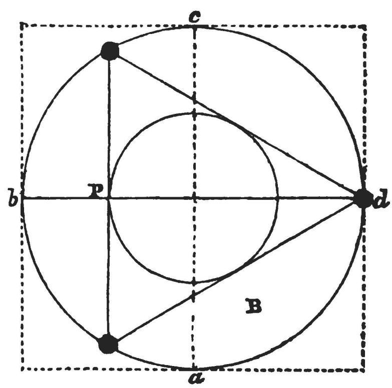
PLATE XXIII.



one and one-third of the primary magnitude which is made the standard of measure, and (proposition I., chapter ii.), it is seen, that the circle *inscribed* in the triangle as above, forms the basis of the area of that triangle, when it shall be measured by circumference and radius, which are the only legitimate elements of area in all shapes alike.

Again: *Thirdly.* It is seen that the equilateral triangle (Plate XXIV.), whose sides make the distance between these revolving bodies, is an angular shape, and being measured in the usual way of measuring angular shapes,

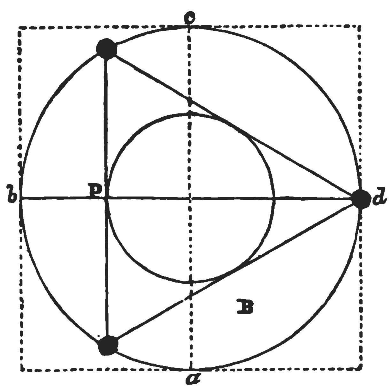
PLATE XXIV.



its area equals the perpendicular, P, *d*, by half the side. Now, let the perpendicular, P, *d*, equal *one.* Then it is seen that the diameter of the circle B, which these bodies describe in a revolution, is one-third greater than the perpendicular. Hence, in performing a complete revolution, these bodies describe a circumference equal to *one* *and one-third* the circumference of *one diameter*. In other words, their relative motion is again seen to be as *four* to *three* of *one primary* circumference.

Fourthly. These bodies, which are revolving together, are known (by hypothesis) to be equal to one another in magnitude, and consequently, equal to one another in all the elements concerned in their revolution.

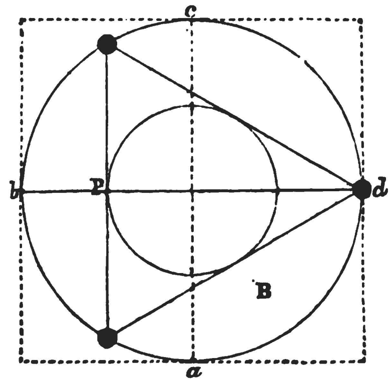
PLATE XXV.



Now, let us suppose that their distance from each other equals *one*. That distance is seen to be the *side* of an equilateral triangle inscribed in the circle B, whose circumference they describe in one complete revolution. (Plate XXV.)

Now, the side of an equilateral triangle inscribed in a circle, equals the perpendicular from the base of an equilateral triangle, *whose side* equals the diameter of the aforesaid circle; and therefore, because the *square* of the side of *any* equilateral triangle, equals one-third added to the *square* of its perpendicular, and because the *square* of the side of the equilateral triangle inscribed in B equals *one*, therefore the *square* of the diameter of B equals *one and one-third*. Hence the area of B equals one and one-third the area of a circle whose diameter is *one.* Hence, in describing the circumference of B, the relative motion of these three revolving bodies shall be as *four to three*, or *one and one-third* the *area* of a circle whose diameter is *one.*

PLATE XXVI.



By proposition XII. chap. ii. it is shown that the true and primary ratio of circumference to diameter of all circles, which can be expressed in whole numbers, is *four times* the *area* of one circle *inscribed* in one *square* for the ratio of circumference, to the *area* of the *circumscribed square*, for a ratio of diameter. Therefore it is evident, that if the circumference of B shall be resolved into such primary parts, as shall express the circumference of *one* diameter in whole numbers, and in its exact relation to area and diameter, without a remainder in either, then the circumference of B, shall equal *one* and *one-third* of one primary circumference, such as may be expressed in whole numbers; because the area of the square *circumscribing* B, equals *one and one-third*, when the side of the equilateral triangle *inscribed in* B equals *one.*

Fifth and lastly. These revolving bodies must be supposed to revolve upon a value, in which diameter and area form exact and equal portions, and the only circle in nature whose diameter and area are equal to one another, and identical in numbers, is a circle whose circumference is *four*; hence the relative motion of three bodies of equal magnitude, revolving together, cannot be otherwise than one and one-third of *such parts*.

It is evident from all the foregoing demonstrations, that, if we suppose the elements of which this problem is composed, to be *magnitudes*, and take *them* as a standard of measure, whether such magnitudes shall be equal portions of the *area* of a circle, or of its *circumference*, or of the *square* of its *diameter*, or whether we take as our standard of measure, the *distance between* these revolving bodies, which makes the *side* of a triangle, or the *perpendicular* of such triangle, or its *inscribed circle*, in all cases, and in every case, the relative motion of these three revolving bodies *must be* as *four to three*, or *one and one-third* of *such magnitude as is made the standard of measure*, and there is no other standard of measure which can be mathematically assumed in the premises, which I have not here considered.

The proposition is therefore demonstrated, that three gravitating bodies of *equal magnitude*, revolving together, their relative motion *shall be* as *four to three*, or *one and one-third* of *one primary* circumference!!

It will be obvious to any one, that in the foregoing demonstration, I have assumed, that the *magnitudes* of the revolving bodies are all *equal* to *one another*, and hence their forces, distances, and velocities, are all *equal to one another;* consequently, they all revolve on the same circumference, as shown in the several Plates, from XXII. to XXVI.; therefore, they cannot revolve *about* each other, but must *follow each other* round a common center. But in the problem of the revolution of the moon about the earth, and the earth and the moon together about the sun, the magnitudes are *all unequal*, and hence their distances from each other, their forces and velocities, are all *unequal*, and they are known *not to follow each other*, as in the foregoing demonstration, but to revolve about each other in the order above stated.

It may, perhaps, therefore be inferred that the foregoing demonstration is not applicable to *such gravitating bodies*. But it must be observed, also, that the EQUALIZING POWER of all the elements of the problem, *are in full force and operation here*, as well as in the problem just solved, and that the chosen distances, forces and velocities, are in exact proportion to the relative magnitudes of the bodies revolving; and hence their *relative* motion shall be *still the same*, with this difference only, that because the moon revolves about the earth, and the earth and the moon together revolve about the sun, *therefore* their relative motions being expressed by time (which is also relative), the following proportions ensue:

First proportion. As *one primary circumference of a circle is to the moon's time about the earth, so is the moon's time about the earth to the earth's time about the sun!* (See the practical application in propositions II. and III., which follow in this chapter.) It must be borne in mind, however, that in the above proportion, reference is had to the revolution of the earth and the moon over the value of a *complete circle*, and *not* to the full sidereal lunation or mean year, each of which are *greater than one circle.* (See introduction to this chapter.) Also, the time here meant is *circular time*, or one revolution of the earth in space. (See table of time in the introduction to this chapter.) It must also be borne in mind, that in the above proportion, reference is had only to the *relations of decimal numbers*, and no reference is made to any geometrical standard of measure in the revolutions of either body. But because *magnitude* is the controlling element in the problem, with the power of equalizing all the rest, therefore, the first proportion as above given being true, a *second proportion follows*, which is strictly geometrical in its character, and which makes the whole definite. It is as follows: *The square of the diameter of the moon is to the square of the diameter of the earth, as the moon's time round the earth, is to the earth's time* *round the sun,—the time here meant being circular time, as before.*

The calculations showing the method and the result of these proportions, will be found in prop. IV. and V. which follow in this chapter.

The true and *primary* ratio of circumference to diameter of all circles, as shown by the twelve propositions of chapter ii., is 20612 parts of circumference to 6561 parts of diameter; and by the solution of the foregoing problem it is shown, that the relative motion of three gravitating bodies, as of the moon, the earth and the sun, is as *four to three*, or one and one-third of one *primary circumference.*

With the solution of these two problems, and keeping in mind the preliminary remarks made at the commencement of this chapter, we are prepared on simple, original, and *mechanical* principles, to reduce to numbers, the great circles performed by the revolution of these gravitating bodies.

The *first*, and to us relatively, the *primary* orbit of nature, which is fulfilled by these revolving bodies, is a sidereal lunation, or the passage of the moon round the earth from opposite a fixed star to opposite that fixed star again,—first because the motion is directly around our earth, and secondly, because the fixed star so-called, has the least apparent or to us relative motion, of all the heavenly bodies; therefore,

PROPOSITION II.

The moon's orbit (or moon's time) round the earth in a sidereal lunation, over the value of a complete circle, is one and one-third of one primary circumference.

The ratio of circumference and diameter being 20612 to 6561, the moon's orbit =20612×1⅓ =27.48266666+, which pointing off 2 figures to the left for days, (because the first figure or left hand unit of diameter being 6 (6,561) therefore circumference has two left hand places of units) I say is the exact time of the passage of the moon round the earth, over the value of a complete circle, the time being in circular days of 23h. 51′ 23″ 20‴ each, and therefore 27.48266666+ ×5153000‴ (the value of 1 circular day) =141618181.3333+ ÷5184000‴ (the value of 1 solar day) equals 27.3183220164+ which reduced to the proper divisions of solar time =27d. 7h. 38′ 23″ 1‴ 20⁗, which I say as before is the exact time of the passage of the moon round the earth over the value of a complete circle.

But because, as has been shown, all natural periods of time are *greater* than one circle, and because a sidereal day, or the revolution of the earth on her axis from opposite a fixed star to opposite that fixed star again, is greater than a *circular day* or one revolution of the earth on her axis *in space*, the difference being 4′ 40″ 46‴, and because by the moon's passage round the earth she gains or performs one motion of the earth, *therefore* the difference between one *circular day* and one sidereal day is to be added to the moon's motion to complete the lunation, and therefore 27d. 7h. 38′ 23″ 1‴ 20⁗ +4′ 40″ 46‴, —27d. 7h. 43′ 3″ 47‴ 20⁗, which I say is the exact period of a sidereal lunation, the only error being the astronomical error in the length of one sidereal day (in adding the difference 4′ 40″ 46‴), which by long observation is known to be less (much less) than one-tenth of one second.* The error therefore is less (much less) than one-tenth of one second of time in a lunar month!!

When I say that the above is the exact period of a sidereal lunation, I must be understood that it is the *mean period* which the moon observes through all time. Whether the moon's motion is or is not sometimes accelerated for a long period and again diminished as much, does not touch this demonstration, but is a question standing by itself. It should be observed, however, that the period of a sidereal lunation as given by me above, is nearly *one-fifth of a second* in a lunar month, less than

* It is less than one hundredth part of one second.

the period given in astronomical time; and the difference applied to the time of eclipses which happened before Christ, is found to agree exactly with the amount of acceleration which the moon is supposed to have received in the last two thousand years; and hence my period is known to be the true period of the moon's motion round the earth.

The second orbit fulfilled by these three revolving bodies, is that of the earth and the moon together, revolving about the sun. This revolution, in consequence of all natural periods of time being greater than one circle, becomes a little more complex than a lunation. According to the main proposition of three gravitating bodies, however, it proceeds upon the same principles and is equally mechanical, but with this difference, that instead of the sun passing round our own earth as the moon actually does, the earth as the centre of the moon's orbit moves round the sun carrying the moon with her, while the sun *appears* to move round the earth and the moon together in the same period of time.

In the solution of the problem of three gravitating bodies I have shown the existence of the following proportion between the motion of the earth and the moon, viz., "That as one primary circumference of a circle, is to " the moon's time round the earth over the value of a "complete circle in space, so is the moon's time round the "earth to the earth's time round the sun over the value "of a complete circle in space." Hence, in calculating the earth's orbit round the sun by the relative motion of three gravitating bodies, we must take the moon's orbit (or moon's time) as our primary circumference, or standard of measure; therefore,

PROPOSITION III.

The earth's time round the sun over the value of a complete circle in space is as four to three, *or* one and one-third *the moon's time round the earth over the value of a complete circle in space.*

Therefore the moon's time round the earth, being 27.48266666+, therefore the earth's time round the sun =2748266666+ ×1⅓=366.4355555+, which pointing off *three* figures to the left for units, for the same reason that two figures are pointed off in a lunation, viz., because diameter has become such that circumference has three places of units, *I say*, is the exact time of the earth's motion round the sun over the value of a complete circle in space, the time being in circular days of 23h. 51′ 23″ 20‴ each, therefore 366.4355555+ ×5153000‴ (the value of one circular day) =188824211.77777+ ÷5184000‴ (the value of 1 solar day) =364.244293552+ which being reduced to the proper divisions of solar time

—364d. 5h. 51′ 46″ 57‴ 46⁗. But because a sidereal day is greater than the mean between a *circular* day and a solar day, by an excess of 44″ 52‴ (see table of time, page 98), which difference necessarily belongs to the sun's motion in his relative position to the earth and a fixed star so-called, therefore we are to add the above period 44″ 52‴ ×1⅓—59″ 49‴ 20⁗ and 364d. 5h. 51′ 46″ 57‴ 46⁗+59″ 49‴ 20⁗—364d. 5h. 52′ 46″ 47‴ 6⁗. Now also because the earth in passing round the sun from opposite a fixed star to opposite that star again, gains one revolution on her axis from opposite that star to opposite that star again, or one sidereal day, *therefore*, we are to add to the above period one sidereal day to complete the mean year. Therefore 364d. 5h. 52′ 46″ 47‴ 6⁗+23h. 56′ 4″ 6‴—365d. 5h. 48′ 50″ 53‴ 6⁗, which I say is exactly the period of the mean year, the only errors being 2⅓ the amount of error in the astronomical time of the length of one sidereal day, viz., once in adding the sidereal day gained by the earth's motion round the sun, and once and one-third in adding the difference, 44″ 52‴, ×1⅓. And as such error is known to be less (much less) than one-tenth of one second of time, therefore the sum of the errors in the above period are less (much less) than two-tenths of one second in the mean year!!

If from the above period of the mean year, we deduct the excess of difference between one circular and one sidereal day, and one sidereal and one solar day, viz. 44″ 52‴; or, if instead of adding one sidereal day for the motion which the earth gains in passing round the sun, we add *one circular* day and the difference between one sidereal and one solar day, we shall then have the period of the solstitial or solar year, viz. 365d. 5h. 48′ 6″ 1‴ 6⁗, which is the known truth within the smallest appreciable divisions of time. This also is a mechanical necessity of the whole principles advanced, because as it will be seen that the earth gains exactly one motion on her axis while performing the value of a complete circle in space while passing round the sun, therefore the amount of the sun's precession of a fixed star shall exactly equal the excess of difference between a circular and sidereal and a sidereal and solar day.

I have several methods of determining from the above periods the amount of a solar lunation, or the moon's synodical period, and by every method I find the exact period to be 29d. 12h. 44′ 2″ 50‴ 31⁗, which also agrees with the most exact time of her conjunctions as observed by astronomers for any number of centuries past.

I have thus, without the use or help of observations of any kind, but with the aid only of the solution of the Quadrature and the problem of three gravitating bodies, and operating only with the simple properties of numbers and with perfectly mechanical principles, determined what the periods of *four* great astronomical circles SHALL BE, viz. a sidereal and a solar lunation, the mean year and the solar year. And these periods, though differing very minutely from astronomical time as given, are found, nevertheless, upon examination, to agree exactly with the conjunctions of nature without correction or allowance. If the same calculations be made by geometers' approximate ratio of circumference to diameter, the result is more than one second of time in a lunar month and near fifteen seconds in the mean year *less than the known truth*, and will agree with no natural period of time whatever. If therefore the solution of the problem of three gravitating bodies *be true*, then it is certain that the geometers' ratio of circumference to diameter is *not true*, but that it is less than the truth, as I have demonstrated both by inverse and direct propositions, chap. i. and ii. And if the solution of the problem of three gravitating bodies *is true*, then it is equally certain that my ratio of circumference and diameter *is true also*, because it agrees with the truth of nature, in the revolution of these great circles. It alters nothing, and matters not in the least, that these revolving bodies do not move exactly in circular orbits, because by the well known law, that they "describe equal areas in equal times," their orbit motion is made *exactly equal to a* *circle.* If therefore geometers admit the solution of the problem of gravitating bodies, they must admit my ratio of circumference and diameter *also*, or they must deny the truth of these astronomical periods, which are established by long and the minutest possible observations of time as shown by the conjunctions of nature. And if they reject my ratio, they must disprove both the quadrature and the problem of gravitating bodies, neither of which can they do, by any show of reason, which cannot be proved to be unsound and false.

It must be observed, that astronomers have arrived at their great accuracy in time, not by a simple mathematical problem, as I have done, nor by the geometrical accuracy of their ratio of circumference and diameter; but at a great expense of time, money and labor, through a long period of centuries, by taking a great number of careful observations at remote times and distances from each other, then by comparing the whole together, and taking the mean of all the differences for the truth; and by thus taking the mean of all their errors for the truth, they arrive at an accuracy in the computation of time, which their ratio of circumference and diameter cannot give them; not however, without liability to some amount of remaining error. But my system requires no such outlay of time, money or labor, nor does it claim the indulgence of a correction of errors. It proceeds only upon the simple properties of numbers, and principles of mechanics, and points out to us not merely what the truth *is*, but what the truth necessarily *shall be*, and gives a reason for it, why it shall be so, and cannot be otherwise.

It will *not* be understood, however, that the foregoing is the limit of the application which may be made of the quadrature to the astronomical circles. I have already applied it *extensively, and with success*, to others of the most important problems in astronomical science; and my present judgment is satisfied, that it is capable of being applied, *ad infinitum*, to new discoveries of the laws and combinations which enter into the system of the universe, even (if the mind of man could embrace so much) to determining the time, distance, magnitude and motion of every revolving body within the range of telescopic observation. It will contradict *no known law of nature.* It will confirm the truth of Kepler's law, and the law of gravitation, as discovered and principally explained by Newton; but it will *not* confirm all else that Newton has said on kindred subjects. And if my present judgment is not mistaken, these problems, when understood and received, will, by the simplest possible *mechanical evidence*, put to silence and to rest forever, some of the stupendous theories which have occupied the reasoning of men's minds for centuries, and which have been *received, only*, because they emanated from great minds, and *believed, only*, because no one knew to the contrary. It is not my intention, however, at present to make a show of mathematical curiosities or astronomical wonders; I desire only to prove the soundness of the basis on which I reason, by the well known truths of nature, in which there can be no mistake, leaving the further discovery of new truths, and the correction of old errors, to future development.

The periods of time as given by me in the foregoing problems, and astronomical time as given by the best authorities, will stand as follows:

A Sidereal Lunation.

Astronomical time.	My period.
27d. 7h. 43′ 4″.	27d. 7h. 43′ 3″ 47‴ 20⁗

Solar Lunation.

Astronomical time as commonly given. 29d. 12h. 44′ 3″. The synodical period as given by M'Kay, the English navigator. 29d. 12h. 44′ $2\frac{8}{10}$, or 29d. 12h. 44′ 2″ 48‴	My period. 29d. 12h. 44′ 2″ $\frac{84+}{100}$. or 29d. 12h. 44′ 2″ 50‴ 31⁗

Mean Year.

Astronomical time as given by the best authorities thirty years since. 365d. 5h. 48′ 49″. As given by the latest and now esteemed the most accurate authorities, taken from a work of Dr. Dick's. 365d. 5h. 48′ 51″	My period. 365d. 5h. 48′ 50″ 53‴ 6⁗

Solar Year.

Astronomical time. 365d. 5h. 48′ 6″.	My period. 365d. 5h. 48′ 6″ 1‴ 6⁗.

From the above comparison it will be seen that my period differs from M'Kay's but the *thirtieth* part of a second in a mean solar lunation, or lunar month; from Dr. Dick's but the *tenth* part of a second in the mean year, and in the solar year from astronomical time, but the *fiftieth* part of one second of time; and it is known as I have already said, that astronomers, notwithstanding their great accuracy, are still liable to some very small remaining error, and it is admitted, that because time is the infinite division of motion, therefore no two relative periods can be stated *exactly;* yet the differences in all the above cases, are portions of time so incomprehensibly small that no one can estimate them.

The periods of time esteemed to be correct thirty years ago, are those first given in the foregoing table, viz.:

A sidereal lunation, 27d. 7h. 43′ 4″.

A solar lunation, 29d. 12h. 44′ 3″.

A mean year, 365d. 5h. 48′ 49″.

And on these I am induced to believe, most of the astronomical tables in use were calculated. It will be seen that my periods differ from these a trifle more than from those established by the more recent surveys, being *less* than either lunation by nearly *one-fifth* of a second, and in the same proportion *greater* in the mean year, or nearly two seconds in the year. Within the last thirty years, however, new sets of observations have been made in Europe, and a new deduction made of the mean year, making it as before quoted from Dr. Dick's work, 365d. 5h. 48′ 51″, thus agreeing with *my period* within the *tenth part* of a second in the year;* but I am not aware whether any correction has been made in the time formerly given for the lunations. It can, however, be shown, I think, that in the assembled motion of any number of revolving bodies, as of the planetary system, if the period of any one of them be fixed *too small*, ob-

* It is worthy of remark that I was not aware of this correction of the mean year which makes it agree with my period, until within the past year, and several years after I had made all my calculations.

servation will show the period of any other one to be *too great* to meet the conjunctions which nature makes, and to make the conjunction by calculation, that body whose period is given too small must be retarded or its period made greater, and the other must be accelerated or its period made less, it being a *mean* of *both their motions* which brings them into line. It is self-evident, therefore, that in changing the astronomical period of the mean year as before stated, making it *greater* by nearly two seconds than formerly, astronomers should have changed the periods of the lunations also, making them proportionably *less* than formerly, or otherwise the two (the earth and the moon) will not perfectly conjoin according to the calculations made on their respective periods; because, as before remarked, it is a *mean* of *both their motions* which brings them into line. And if *this be done*, then the periods of the lunations last above given will be made to agree with *my periods*, to the sixtieth part of a second of time, and they will also agree within the thirtieth part of a second with the synodical period or mean solar lunation as quoted from M'Kay in the foregoing table, which has long been supposed to be the nearest possible approach to the mean time of the conjunction of the earth, sun, and moon.

Let it here be noticed that the mean of the difference, which my period of the mean year is *greater*, and the lunations are *less* than the periods formerly in use, is exactly equal to the supposed acceleration which astronomers say the moon has received in the last two thousand years or more; and this difference being applied by time to the relative motion of the earth and the moon, will place the two in conjunction with the sun, just at the time and place where an eclipse is known to have happened more than four hundred years before Christ, whereas the astronomical tables will be fully an hour and a half out of the way. It is, therefore, certain that my periods are, to the smallest appreciable divisions of time, the *true periods* of the mean *relative* motion of the earth, sun and moon, and the inference from all these truths, is a perfect and practical demonstration of the truth of my ratio of circumference and diameter. If these be facts, they seem to contradict the existence of the acceleration of the moon, such as is supposed by astronomers to exist, and to imply instead a perfectly equable mean motion of that body.—It is not my intention at present to argue the points of acceleration, or *no acceleration.* I do not say at present that acceleration is impossible; because nothing is impossible with the infinite wisdom and purpose which governs the universe; *but I do not credit it*, notwithstanding the high authority on which it is asserted, because, *First*, it is a notion which grew up only on the discovery of error in the calculation of eclipses through long periods of time, and in efforts to account for that error. *Secondly*, all the appearances which indicate it may be easily accounted for, without the necessity of its existence, and nothing exists in nature which is *unnecessary*. *Thirdly*, it indicates defection in nature's laws, and therefore it cannot be true.

In the solution of the problem of gravitating bodies I have established certain *proportions*, which, in order not to be misunderstood, it is proper to reduce to calculations; therefore,

PROPOSITION IV.

First proportion. As one primary circumference of a circle is *to the moon's time about the earth over the value of a complete circle in space*, so is the moon's time round the earth, *to the earth's time round the sun over the value of a complete circle in space.*

It will be evident that any two parts of the above proportion being known, the third may be found. By proposition III. (this chapter) it is shown that the moon's time round the earth is, to one primary circumference, as 27.4826666+ to 20.612; therefore,

: 20612 :: 274826666+ : 274826666+—366.435555+

The time in the above periods, is *circular time*, and pointing off two figures to the left in the first period, and three in the second for units, on reference to propositions II. and III. it will be seen how these periods are brought to the *solar time* contained in a sidereal lunation and the mean year.

The second proportion, in the problem of gravitating bodies, is a geometrical proportion, as follows:

PROPOSITION V.

Second proportion. The *square of the diameter of the moon* is to the *square of the diameter of the earth*, as the *moon's time round the earth* over the value of a complete circle in space, is to the *earth's time round the sun* over the value of a complete circle in space.

Three parts of the above proportion being known, the fourth may be found; therefore let the moon's diameter be the part unknown. We have already seen that the moon's time —27.4826666+ and the earth's time —366.435555+ (circular time), and the diameter of the earth has been ascertained by actual measure to be 7,912 miles, which it no doubt is, very nearly. Admitting then, that the earth's diameter is 7912 miles, then the square of her diameter —62599744, therefore : 366.435555+ :: 62599744 : 27.4826666+ —4694980.8+ and : $\sqrt{4694980.8+}$ —2166$\frac{7}{10}$+ which, I say, is the true diameter of the moon, and neither one mile, or tenth of a mile, more nor less, the only condition being, that the diameter of the earth is 7912 miles.

Dr. Bowditch gives the moon's diameter as 2161 miles, and others have given it at 2180. It will be seen that the diameter proved by me, is nearly 6 miles greater than that given by Dr. Bowditch, and 13 miles less than is given by some others. But the fact that astronomers differ at all, proves their method to be imperfect, and consequently liable to error, sometimes greater and sometimes less, while the close approximation on each side is a very strong argument in favor of the truth of my proportion, even if it were not here seen to be accurately deduced from mathematical principles.

In my introduction to the Quadrature (chapter ii.), I there signified that my course of reasoning would be strictly original, and wholly independent of any arbitrary rule,—*perfectly conformable to nature*, yet not confined to the rules of art; and I recall attention to this remark, because it is necessary to be borne in mind by those who may undertake an examination of the subjects treated of in the present chapter.

In respect to the astronomical circles, it must be observed, that the manner in which I have treated them embraces no other facts or principles, than the simple relations between numbers, shapes and motion, and no reference is made, or intended to be made, except in the fifth proposition, to either magnitudes or distances. In treating of the astronomical circles, therefore, I have simply treated them each as *one circle*, made up of the parts which compose the *primary relations* between the circle and the square, but without any reference to any standard of measure in art.

It is to be regretted, I think, for the sake of science, that so little examination has been made into the recesses of nature, to supply us with standards of measure. So far as I know, science has given us no *natural standards*. The French standard, derived from the measurement of an arc of the meridian, is but an imperfect attempt. We are told, in English, that "three barleycorns make one inch," and the length of three barleycorns which grew in the time of one of the English kings, seems to be the only contribution which nature has been called upon to make, to supply us with standards of measure. It is not at all wonderful, therefore, that such standards have no applicability to time, as created by the motion of revolving bodies, or to any of the fixed laws of nature whatever.

It has been objected by some caviling minds, that calculations like these astronomical circles, which are based only on the properties of numbers, or of shapes, but which have no standard of value, cannot be of any practical use. But this is a very short-sighted objection, and to make them of practical application and use, we have nothing to do, but, just as is necessary in any other case, to erect a standard of value in *our own minds*. But in order to make any standard available to us here for any intelligible purpose, it is necessary that it be selected *from nature*, and it must also be a *fixed fact* in nature, and not an accidental truth, like the childish conception of the length of "three barleycorns." To answer the objections made, and to illustrate and prove this position, I am induced to add *another practical question*, which was not intended in the commencement of this chapter.

It is well known that the United States have lately expended a large sum of money for the erection of an observatory in southern latitude, for the purpose of co-operating with others at the north, in determining the sun's distance from the earth; and my purpose is now to show, that this truth may be determined with much greater precision by my principles of reasoning, than by any other method, and *without the help of observations of any kind*. As there is an uncertainty with astronomers, at least to the extent of several millions of miles, what the sun's mean distance really is, it may not be uninteresting to compare the results of my principles of reasoning with the actual observations, when they shall be completed. Therefore—

PROPOSITION VI.

The mean distance of the sun's center from the center of the earth, or that at which the earth would revolve, if the area or plane of her elliptical orbit were made the area of a circle, is *eleven thousand six hundred and sixty-four* diameters of the earth, neither more nor less; admitting, therefore, that the earth's diameter is 7912 English miles (which it no doubt is pretty nearly), then the sun's center is distant from the earth's center as above 92,285,568 English miles, and neither one mile more or less.

In following out the above proposition to demonstration, in order to make the connection of my principles of reasoning clear and manifest to the perceptions of others, it is necessary here to lay down as axioms certain truths which have been proved.

First. The circle is the basis or beginning of all magnitude or area. (Proposition III., chapter ii.)

Secondly. Any expression of numbers in relation to material things is also an expression of magnitude. (Proposition VI., Appendix.)

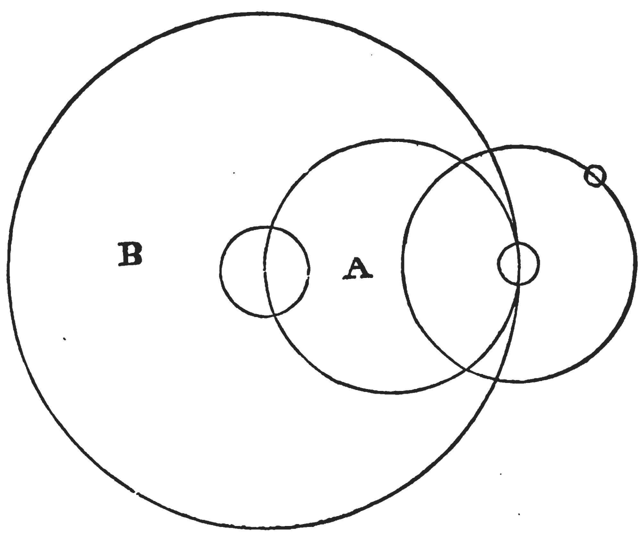
Third. A point is therefore a magnitude when considered as *one.* (Proposition VI., Appendix.)

Fourth. A point in reference to space or extension on all sides of it, is therefore a molecule or globe, and in reference to a plane, it is a circle. (Proposition VI., Appendix.)

A point, when considered and treated as *one*, is therefore the least possible existence of magnitude,—the original principle of matter; a *fixed fact* in nature, unchangeable, imperishable, and such, that the union of many points according to their chemical affinities, becomes matter developed to the senses, and because we can have no other comprehension of the development of matter, therefore, relatively to us, this is an *absolute truth.* A point is therefore *such*, that it has an *exact relation* to *every development of matter* in *our world* and its *atmosphere;* therefore, if the magnitude of a point were a thing within our comprehension and grasp, *it would form a perfect standard* of *measure*, and by enumerating points beginning with one, and counting upward, numbers would at length express the magnitude of our world; and in the process of counting, we shall have enumerated the exact *relative* magnitude, one to another, of *everything contained in it.* A point is therefore a *perfect standard of measure*, and *any number of points* is a perfect standard of measure for any *greater number* of points. Hence our earth being a magnitude made up of points, and a *fixed fact in nature*, is therefore a perfect standard of measure for all greater magnitudes that surround it.

When we attempt to comprehend or to estimate the distance from our earth to the sun, we enter on a higher order of creation, and mentally pass from the contemplation of things in the world, to things in the universe, where *worlds are points*, bearing exactly the same relation to the *infinite whole*, which the incomprehensible and undeveloped point bears to our world, because each runs to *infinity*, and because a point is *one*, and therefore emphatically *the one* to which all other and greater magnitudes are *exactly related;* therefore, let the earth be *one*, and let that be the standard by which to measure the sun's distance.

PLATE XXVII.



By proposition I., this chapter, I have shown that the relative motion of three gravitating bodies, as of the earth, the sun, and the moon, is as *four to three* of *one primary circumference* of a circle; but this, as has been seen, is without reference to any definite standard of measure. By relative motion, is meant, of course, the *relative change of position* of one body to another. By that proposition, therefore, the measure of a year is the measure of a circle in which the earth and the sun change their relative position, and return to that position again. And by the same proposition, and proposition III., this chapter, it has been shown, that the time in which the earth performs the value of a complete circle in space, being reduced to circumference, it has a diameter of 11664 parts of that which I say is *one primary circumference* in nature, viz., $6561 \times 1\frac{1}{3} = 8748, \times 1\frac{1}{3} = 11664$. It will be evident, on reference to the illustration on the last page (Plate XXVII.), that a circle which is the measure of the *relative change of position* of two of these gravitating bodies, and around which they move *relatively to one another*, is the circle A, whose circumference passes through the center of each body (the earth and the sun), hence the circle A is *the circle* whose circumference I have measured in determining the mean year in proposition III., this chapter, and whose diameter is 11664; in which proposition, I have also made the earth (by her revolution on her axis) to be the *unit* or *stand-*

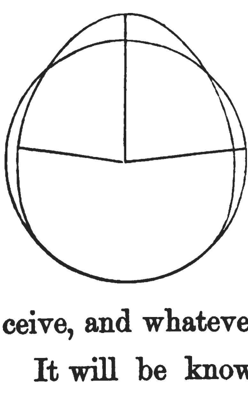
ard of measure. It will be seen, also, that the diameter of the circle A is the radius of the circle B, which the *earth shall describe in passing round the sun*, and therefore the diameter of the circle A *is the earth's distance from the sun.*

Now, therefore, because the earth is a *primary* magnitude, a *fixed fact* in nature, and a *point in the universe whose value is one*, bearing the same relative value in the order of creation, to things in the universe, which the undeveloped point bears to things in the earth, and is therefore a *perfect standard of measure*,—and because she is *herself* the unit or standard of measure which by her revolution determines the value of the circle A in measuring the mean year, and is also, by hypothesis, here made the *unit for determining her distance*, and because the diameter of the circle A is 11664 parts of the diameter of one primary circumference, of which the earth is but *one part*, therefore, the earth's mean distance from the sun, from center to center, is 11664 diameters of the earth, *neither more nor less;* and therefore, admitting that the earth's diameter is 7912 English miles (which it is pretty nearly), then 11664 × 7912 — 92285568 miles, which is the earth's mean distance from the sun, and *not one mile more or less.*

The proposition is therefore demonstrated!!

In order not to be misunderstood in respect to the result in the foregoing demonstration, it is proper for me here to add some explanation of the difference between that which I have called the *sun's mean distance*, and that which is commonly understood by astronomers as such. It will be self-evident to all, I think, that admitting my demonstration to be true, the distance shown is *that* at which the earth *would* revolve in a perfect circle, if the sun *were fixed in the center*; and if this be the fact, then it is equally evident, I think, that the distance shown is the radius of a circle whose area is exactly equal to the plane of the earth's supposed elliptical orbit; because, it is self-evident, that if the earth shall move through an elliptical orbit by an *unequal* motion, *passing over equal areas in equal times*, it is precisely the same thing as passing over the circumference of a perfect circle having the *same area as the ellipse* by an *equal motion*, in exactly the *same period of time*. It will be seen from

PLATE XXVIII.



the illustration (Plate XXVIII.), that the ellipse and the circle having the same area, the radius of the circle is *greater* than the *least*, and less than the greatest radius of the ellipse; and this will be true, whatever elongation the ellipse may receive, and whatever center may be taken as *the center*.

It will be known, also, from the laws which govern these shapes, that the difference between the radius of the circle and the least radius of the ellipse, is *less than half the difference* between the least and greatest radius of the ellipse; therefore, if the sun's mean distance be taken to be half way between the least and the greatest radius of the ellipse, it will be greater than the distance which my demonstration shows; and if the sun's mean distance be taken to be the mean of the *squares* of the two radii of the ellipse, then the distance will be greater still; the latter, I believe to be the mean which is mostly adopted by astronomers; but in either case, it will be seen, that *any distance* shown by them, even if measured with perfect accuracy, will be greater than mine. The angle of parallax, as deduced from the last transit of Venus, is given in Vose's Astronomy, as from the best authorities, as 8″.52 at the sun's greatest distance, and 8″.65 at the sun's *mean distance*,—this latter would give a radius of about 94,300,000+ miles as the mean distance. La Place, who has been esteemed the most accurate authority in these things, thought that the deductions made from this transit were within *one eighty-seventh* of the truth, more or less, he could not tell which; thus leaving an uncertainty of considerably more than *two millions of miles*;—deduct this uncertainty from the distance given above, and with a very moderate allowance for the difference of mean intended, the sum will very closely approximate to that which I say is the *exact distance* at which the earth would revolve in a perfect circle, if the sun were fixed in the center, and the area of that circle is *exactly equal* to the plane of the earth's elliptical orbit, *as she moves at present.* The only qualification to this is, that the earth has a diameter of 7912 miles, neither more nor less. But putting aside *all qualification*, to make the thing *perfectly accurate*, I say, that the sun's distance at the mean, as given by me, is 92,819,114+ of those parts, of which the circumference of the earth is exactly 25,000, and its diameter 7957+. And I say, moreover, that these are the *true parts* at which the circumference and diameter of the earth *should be considered*, according to the French standard of measure, which takes the circumference of the earth as *one*, and proceeds by decimation to fix the value of smaller measures.

It is known, that in consequence of the elliptical form of the earth's orbit, she must move faster in one part of it than in the opposite part. It is known, also, that all observations of the sun, or any of the heavenly bodies, taken from a position on the earth's surface, are liable to more or less error, from the fact, that the earth is *at all times* in rapid motion through her orbit, and on her axis. Hence, if two sets of observations be taken; one, when she is in the largest part of her orbit, and the other, when she is in the smallest, the errors in calculation, arising from the earth's motion, will be greater in one case than in the other, because the earth moves faster in one part of her orbit than in the other. But if the mean distance be taken, at which the earth would revolve in a perfect circle, having the same area as the ellipse, and with a perfectly equal motion, this liability to greater error at one time than another, will be corrected. I do not hesitate, therefore, in declaring, that the mean distance, as shown by me, is the most *accurate*, as well as the most convenient, for all astronomical calculations made from observations, even if any other distance *could be* accurately determined, which it *cannot be* by any method adopted by astronomers, without an uncertainty of considerably more than two millions of miles.

Having thus determined with accuracy, the mean distance from the sun, at which the earth would revolve in a circle having the same area as the ellipse, by Kepler's law, that "the squares of the times are as the cubes of the distances," we have a *correct basis* on which to determine the mean distance from the sun, *of every planet and satellite in the solar system*, a thing never before attained. And the only question for astronomers to decide, is, is my demonstration true to the operations of nature, according to the principles set forth in it? I affirm that it is, to the smallest fraction, and challenge them to the disproof by any means in their power, which is not liable to error equal to the disagreement which they may find!!

NOTE.—In a series of Lectures on Astronomy, &c., lately delivered at the Hope Chapel in this city, and at Brooklyn, by a learned professor from the West, the professor was understood to say, "that the construction of "the heavens was ordered for the exercise of man's rea-"son,—that the Creator *might have made* things much "more simple, but then man would not have had scope "for his reason. For instance," said the professor, "the "Creator might have made the planets to revolve in per-"fect circles, and then *any one* could have calculated "their motions." These remarks struck me at the time as rather singular, because it is self-evident that the *unequal motion* of the planets over an ellipse, passing over equal areas in equal times, is precisely the same thing as passing over a circle by an *equal* motion in exactly the same time. And moreover, the professor must have known, if he remembered his mathematics at all, that if the orbits had been circles, those were precisely the things above all others, which *he*, at least, could *not calculate*. The benefit of popular lectures on such subjects cannot be doubted, but the disposition of learned men to embellish truth, and attempt to instruct the Deity on such occasions, is sadly to be regretted.

APPENDIX.

Many persons, I think, imagine, that the Quadrature of the Circle is only a kind of mathematical puzzle, which if ever solved, some one should at length work out by a single proposition; and few, perhaps, will be prepared to believe that a work so large as this has already become, is really necessary in order to demonstrate satisfactorily any single truth. But such persons, I think, can have very little idea of the numerous ramifications into which mathematical science has extended itself, and how intimately it is associated, not only with *every other* practical science, but with every material truth in the known world.

If the ratio of circumference to diameter had been among the early discoveries made, and the whole superstructure of mathematical science been built upon the knowledge of its truth, it would then have been easy enough to satisfy inquiry by the demonstration of a single proposition; but unfortunately such is not the fact. The foundations of the science were laid without this knowledge; and under the guidance of multitudes of the most acute minds the world has produced in every age, it has extended itself seemingly in every possible direction, and embraced almost every possible subject, until it must be admitted, that it is at this day, the *most* perfect of *all* the sciences. Yet it is *not perfect* in any of its branches. In geometry especially, the most beautiful and useful of all, there is something yet lacking, and that something lies at the foundation of all truth,—it is the QUADRATURE OF THE CIRCLE, or a knowledge of the exact relations between straight lines and curved lines, which has never yet entered into the structure of the science. The science is, I think, rightly esteemed the most noble, most useful and most beautiful structure in existence, the production of human intellect searching after truth, but even this most perfect production of intellectual labor is not yet perfect. It was begun with a knowledge of only a part of the truth,—without understanding *all* the principles which in its upward progress to its present magnificent proportions would be brought into practice,—and as in all such cases a want of a knowledge of *all* the principles which were to be carried out, has necessarily led to some error;—some of its materials are heterogeneous, and they have become mixed and confused;—some of its proportions are unjust, because not exactly true,—some of its parts will not match, and the workmen have tried to make them match by correcting their differential properties. And we all know well enough that in architecture when we attempt to correct a mistake in this way, instead of pulling it down and building up again, we must go on correcting mistakes forever,—if but one stone is out of place, it will go on displacing others until the whole building is marred,—it is a mathematical result that it should be so, and no power of man can correct it without correcting the first error. Such, then, is the condition of the structure of mathematical science at the present day, and to carry out the figure, the building can never be complete. It wants another and a chief corner-stone to rest upon, before the cap-stone can be laid and the whole present a finish which the Deity himself may look upon without pity on the intelligence of his creatures. And to accomplish this, we must first remove all that part of the superstructure which is out of place, and this is in fact the thing proposed when we attempt the solution of the Quadrature.

To supply this chief corner-stone we must go back to the first error, dislodge it from its foundation, and establish the truth in its place, by determining without condition or qualification the exact relations between straight lines and curved lines; and we must then follow up the first error, through all gradations of the received science, and wherever it has established itself as a principle, we must prove such principle to be false by unmistakable evidence, dislodge it, and take it away entirely; or otherwise it will become the foundation of other false principles through every gradation of an infinite series.

Such is the constitution of the human mind, and such the force of education, that the minds of mathematical professors are, with exceedingly rare exceptions, formed *upon the rules of the written science*, and they are unwilling, and often *unable* to comprehend any other. One highly distinguished among them lately remarked, respecting himself, that these ideas (meaning the received theories of mathematical science) "had become a part "of the furniture of his mind, and were too strongly fixed "to allow him to *consider* any other." From this cause I have found the Professors as a body, though learned in the received theories, to be among the *least competent* to decide on any newly discovered principle. Their interest, education, pride, prejudice, self-love and vanity, all rise in resistance to anything which conflicts with their tenets, or which outruns the limits of their own reasoning. So little do they look beyond the principles inculcated by education, and so tenaciously do they hold on to these, that when driven from one principle they fall back upon another, and when beaten from all, they return again to the first, and maintain themselves by dogged assertion, or by charging their assailants with ignorance and a lack of science; such at least I have found to be the character of the professors, in every approach I have made to them; and this being the case, if I would have my work acknowledged, there must no foothold be left for them to rest upon. I think I shall be justified by the candid judgment of well-informed men when I say, that, in consequence of this character of professors, the practical men of the age are at least a century in advance of the schools, in all useful scientific knowledge. I have made these remarks as a reason and in explanation of the necessity of following out in further minutiæ the errors to which various problems in geometry are subject in consequence of the error in the Quadrature.

In the preceding chapters I have made occasional reference to facts and principles not previously demonstrated, and which, in a work strictly mathematical, or which was designed for practical instruction, should have stood first, as elementary truths, on which subsequent demonstrations were to be based. But to have made my preliminary demonstrations too diffuse, would, I think, have diverted attention from the main object; and I have therefore thought fit, under the head of an Appendix, to demonstrate such propositions as will answer to the above references and sustain the argument.

One of the facts stated as above in the course of this work, but not previously demonstrated, is as follows:

That "the so-called perimeter of the circumscribed "polygon of geometers is *not* a *circumscribed* perimeter, "but that the center of each side of the perimeter *coin-* "*cides* with a part of the area of the circle, and at an "infinite number of sides is brought *wholly within the area of the circle.*"

The first general proposition on which geometers proceed, in approximating to the circumference of a circle, is as follows,—that "the circumference of a circle is greater "than the perimeter of an inscribed polygon, and less "than the perimeter of a circumscribed polygon, what- "ever may be the number of the sides."

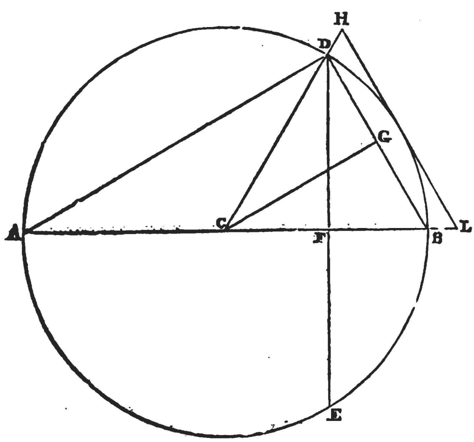
Nothing can be more true than this general proposition,—provided, however, that the *conditions* of the proposition be fully adhered to in the demonstration. In the fifth proposition of the first book of Playfair's Supplement to the Elements of Geometry, he demonstrates that "the area of any circle is equal to the rectangle con- "tained by the semi-diameter and a straight line equal to "half the circumference." This proposition is also true, and Playfair demonstrates it by an *inscribed* and *circumscribed* polygon; but the *conditions* of the demonstration are, that the perimeter of the *circumscribed* polygon lies *outside* of the circle "touching it," and on *this condition*, and on *no other*, is the first above named general proposition true (see prop. v., chap. ii.). It will be seen that if the perimeter of the circumscribed polygon lies *outside* of the circle "touching it," then no part of the perimeter of such polygon can coincide with any part of the area of the circle. My object is now to show that the line approximated by geometers as the circumference of a circle, is a line coinciding with the greatest limit of the area of the circle, and exactly equal to the circle, but not *inclosing* or *containing* it according to the true definition and meaning of circumference (prop. v., chap. ii.),—that this result is produced by bringing the so-called circumscribed perimeter *wholly within* the area of the circle, and that consequently geometers by their method of bisection *do not adhere* to the conditions of the first general proposition, and hence their result is not true in its application to the circumference of the circle. Therefore,

PROPOSITION I.

The line approximated by geometers as the circumference of a circle is a line coinciding *with the greatest limit of the area of the circle, but not* inclosing *or* containing *it.*

I now take the eighth proposition of Playfair's Supplement to the Elements of Geometry, book i. It reads as follows: "The perpendicular drawn from the center of a "circle on the chord of any arch, is a mean proportional "between half of the radius, and the line made up of the "radius and the perpendicular drawn from the center on "the chord of double that arch. And the chord of the "arch is a mean proportional between the diameter and "a line which is the difference between the radius and "the aforesaid perpendicular from the center." This proposition is also true in every particular in respect to an *inscribed polygon*, which forever remains inscribed within the circumference of the circle, and if it could be carried out in bisection without any quantities, being lost in the calculation (which it cannot be), it would constantly approach to a line coinciding with the greatest limit of the area of the circle, but could never equal it, much less inclose it (prop. I., chap. i.).

PLATE XXIX.



In Plate XXIX., we have the same diagram which Playfair uses in his illustration, with the exception that I have added the circumscribed line H L. To reduce the proposition to its value in numbers the proceeding runs thus.

The diameter (A B) being considered as 2, the line D E is the chord of one-third of the circumference; it bisects the radius C B, and since C B—1, therefore C F—.5 and C F is the perpendicular from the center C on the chord D E, and the proposition is, that "the chord "of the arch is a mean proportional between the diam- "eter and a line which is the difference between the "radius and the aforesaid perpendicular from the center." Therefore C B—C F—F B, and F B×A B—D B^2, and $\sqrt{D B^2}$—D B, which is the chord of the arch of one-sixth of the circumference, or double the number of sides of D E. And in like manner he proceeds to a greater number of sides.

Now, the circumscribed line H L, according to Playfair's method, is a proportion to D B, as C B is to C G. The chord D B is supposed by geometers to be a line *wholly without* breadth; consequently, it is a line, the center of which exactly coincides with the extreme point of the perpendicular, C G, neither one particle short of it, nor one particle beyond it, the point of the perpendicular itself being, in fact, part of the chord; consequently, the circumscribed line H L, being a proportion to the inscribed line or chord, D B, as C B is to C G, its center (H L) exactly coincides with the extreme point of C G, when C G is produced equal to C B, neither one particle short of it, nor one particle beyond it, so that if the perpendicular C G shall have breadth given to it, then the extreme point of C G, when produced equal to C B, will form part of the line H L. It is evident, therefore, that the perimeter of Playfair and Legendre's so-called circumscribed polygon does *not lie* OUTSIDE of the circle "touching it," according to the conditions of the fifth proposition, book first, of Playfair's Supplement, in which he demonstrates the area of a circle, as before referred to, in this proposition; on the contrary, the perimeter, at the center of each side of his so-called circumscribed polygon, coincides with a part of the area of the circle, and at an infinite number of sides is brought *wholly within* the area of the circle, and therefore, does not inclose or contain it. It is evident, also, that the condition of the first general proposition of Playfair and Legendre, that "the circumference of a "circle is greater than an inscribed polygon, and less "than the circumscribed," is *not adhered to* in the demonstration, and therefore, their result is not true by their own showing, but is less than the truth; because the perimeter of their so-called circumscribed polygon does not lie *outside* of the circle "touching it," according to the required conditions; and because, as has been demonstrated (proposition v., chapter ii.), the true circumference of a circle is a line wholly *outside* of the circle, thoroughly *inclosing* its whole diameter, and containing the whole area of the circle *within* it; therefore, the true circumference of a circle is greater than the *so-called* circumscribed perimeter of Playfair and Legendre, at an infinite number of sides. *The proposition is therefore demonstrated.*

A teacher of mathematics, in one of our institutions, in answer to my second proposition, chapter i., that "the "area of a circle is greater than the area of any polygon "having the same circumference of the circle, whatever "may be the number of the sides of the polygon," writes me as follows:

"You endeavor," says he, "to prove that the polygon "can never equal the circle" (each having the same circumference, and being measured in the same way). "Your reasoning on this appears to be correct; but by "comparing this approximation with some others that "are analogous, I am inclined to believe that it is not "correct. Take, for instance, the series $4 + 2 + 1 + \frac{1}{2}$ "$+ \frac{1}{4} + \frac{1}{8}$, &c. Now, this series will approach to 8, but "can never equal 8; but embraced in an algebraic for- "mula, it can be proved, that it does exactly equal 8, "when the number of the terms are infinite. Let the "series $= x$.

" $x = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, &c., to infinity:

"Then $x - 4 = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, " $+ 2$:

"Then $2x - 8 = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, "

"The last series is identical with the first, and things "which are equal to the same things, are equal to one "another; therefore, $x = 2x - 8$; or $0 = 2x - x -$ "8; or $0 = x - 8$; or $x = 8$."

Now, this "algebraic formula" is, by the learned teacher, called a *demonstration*, showing that my second proposition, chapter i., cannot be true! It looks, on the face of it, almost too ridiculous to be entitled to an answer,—but the learning of the schools must have consideration; and besides, if this proposition be true, then my second proposition is *not true*.

I now desire the reader, therefore, to turn to the second proposition, chapter i., and examine the demonstration which follows; he will see that my demonstration is purely geometrical (not algebraical); the result is a necessity of the immutable laws of numbers,—the reason of that result is palpable to the senses,—the demonstration is therefore accepted as a self-evident truth. Now, what is the character of the *learned teacher's* demonstration? It is an algebraic formula, adopted to prove a thing contrary to the evidence of our senses, and contrary to the operations of numbers; for it is admitted, that in numbers (and numbers are in themselves infinite), the series can never equal 8. It will be seen, that in the treatment of this series by algebraic formula, the conclusion arrived at, or rather assumed, is, that an infinity $= 0$; but I have already promised, in another part of this work, to show that an infinity, whatever it may be, is always such, that in material things it is *capable of increase*, which I shall presently do by this same series. This algebraic formula, then, is called a demonstration; but, in point of fact, it is no demonstration at all. By a demonstration, I understand the making known and certain, something which was before unknown and uncertain. But the whole of the foregoing so-called demonstration by algebraic formula, depends entirely on the assumption or hypothesis—*first*, that infinity $= 0$; and *secondly*, that the series does actually equal 8. But if the assumption or hypothesis be *not true*, then the demonstration is *not true*; and I say, that in this case, the assumption is *not true;* and unless it be first proved by numbers, the algebraic formula proves nothing but what the contrary may be proved by the same formula.

For instance, I will say that the series $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4}$ &c., $= 9$, when the number of the terms are infinite. Let the series $= x$.

$x = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ &c., to infinity, $= 9$;

Then $x - \frac{1}{2}x = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ " $+ 2 = 9$;

Then $2x - x$;

Or $2x - 9 = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ " "

The last series is identical with the first, and things which are equal to the same thing, are equal to one another; therefore, $x = 2x - 9$; or $0 = 2x - x - 9$; or $0 = x - 9$; or $x = 9$.

The absurdity of calling this a demonstration, is, I trust, manifest; yet it is the same which the learned teacher of mathematics, in one of our public institutions, has furnished to disprove my second proposition, chapter i., which is purely a geometrical proposition, geometrically demonstrated. And this is not the only instance which can be found, of the absurd use in the schools, of algebraic formula for demonstrating geometrical propositions,—there are many things thought to be demonstrated, which will not bear criticism. One rule, however, will apply to all such; if the assumption or hypothesis be true, the demonstration is true; but if these be not true, then the demonstration is *not true;* and in this case, I say, it is not true, that the series $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ &c., = 8, or that an infinity = 0, because numbers and things are identical and inseparable, and neither in numbers or things, is there any infinity of division = 0.*

* The same teacher, who so learnedly attempts to refute my second proposition, as above, writes me also in respect to my ratio of circumference, that "it is proved by trigonometry that the length of an arch of 45° to radius = 1;" is equal to a certain series, by which they obtain for circumference 3.1415926+, and hence, he thinks that my ratio cannot be true. And this method, he says, "*is entirely independent of the method of Euclid.*" He would thus argue, it seems, that trigonometry and geometry are two things; and hence the result by what he pleases to call trigonometry, is independent of the result by the geometrical method!! a most potent argument, to be sure. But I trust

The term "infinite," or "infinity," is one often used in mathematics, but no explicit or satisfactory definition has ever been given to it. An infinity, in its fullest sense, whether of magnitude or minuteness, is an incomprehensible term,—no mathematician ever did, or ever can understand it. To suit their own purposes of reasoning, however, mathematicians have assumed that an infinity, or a thing infinitely diminished, equals 0, and therefore, throw it away, as having no appreciable value. Of the error of this course, I have already given an example, in the remarks following proposition IV., chapter i. No schoolboy's mind was probably ever satisfied with this *throwing away* of *infinity*, until, by instruction and habit, he has at length reached the full grown prejudice of his teachers; for he sees, that they are sometimes obliged to reverse the case, and then they endeavor to prove that *nothing* may equal *something!!* If an infinity

it will require no argument to prove that the principles made use of in trigonometry to determine the series, being based only on the properties of straight lines, are precisely the same, and involve the same error as Euclid's method, and therefore, come to the same result. If the method, by a fluxionary series deduced from trigonometry, is right, then Euclid's method is right also, because they come to the same result. But Euclid's method has been proved to be wrong, and to be less than the truth; therefore, the series proved by trigonometry is also wrong. In fact, all such series are nothing but approximations; there is not a single absolute truth in the whole range of them; the very name of an infinite series signifies something which never can be equalled.

be really *nothing*, the term cannot be applied to material things, nor can we reason on the two (finity and infinity), or from one to the other, with the slightest ground of truth for a basis. Therefore—

PROPOSITION II.

An infinity in minuteness is always such, that it is capable of increase; therefore, in material things, an infinity equals one ultimate particle of matter, such, that in the nature of the material or thing under consideration, it cannot be less.

I propose to demonstrate this second proposition by the series $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ &c., and I say that this series, infinitely extended, equals 8, *minus one infinity*, or minus one ultimate particle of matter, such that, in the nature of the matter or thing considered, it cannot be less. The demonstration is by numbers, and it proceeds upon the supposition, that the so-called infinity of mathematics is a point of division *beyond* the power of numbers; therefore, $4 + 2 + 1 + 0.5 + 0.25 + 0.125 + 0.0625 — 7.9375$. I have here carried the division to one-sixteenth only, and it is seen that the sum of the whole is deficient of 8, one part of the last division, and in order to make the sum of the whole equal 8, the last addition must be $\frac{2}{16}$, instead of $\frac{1}{16}$; and the same is seen to be the case, to whatever point in numbers the division may be carried,—two parts of the last division must be added, to make the sum of the whole equal 8; therefore, let the number of divisions be the greatest possible; then, because the number of the terms or divisions is the greatest possible, and two parts of the last division are necessary to be added, in order to make the series equal 8, and by the addition of one part only, an equal part is left; therefore, the part left is infinity, which numbers cannot divide; hence, it is evident that numbers themselves are infinite,—hence the divisions of numbers equal infinity,—hence an infinity equals the greatest possible divisions of numbers,—hence an infinity equals one part of the greatest possible division of any magnitude—hence also, by reciprocity, in the above series, an infinity is such, that by constant doubling, it shall amount to 4; therefore, an infinity is such, that it is *capable of increase. The proposition is therefore demonstrated.*

It will be seen, from the above demonstration, that because *numbers themselves* are thus proved to be *infinite*, as, indeed, our own perceptions tell us they are, and because the series $4 + 2 + 1$ &c., can never be made to equal 8 by *numbers*, therefore, the assumption that the series does actually equal 8, or that an infinity $= 0$, is absurd, and any demonstration by algebraic formula, to the effect that they do, is a pure assumption, contradictory of evidence, and is therefore absolute nonsense.

The general idea which we get of magnitude in the abstract is, that it is something which entirely fills a definite portion of space. Now if we suppose any definite portion of space to be divided first into two equal parts, then four, then eight, and so on to *infinity*, it is evident that the sum of all the parts into which the space is divided is equal to the whole, and one part equals infinity, but if we suppose that infinity — 0, then each part — 0, and the whole space is annihilated, which is absurd; because although we cannot say with certainty that matter cannot be annihilated, we do say and know with certainty that blank and abstract space cannot be annihilated. It is evident, therefore, that *both numbers* and *space* are infinitely divisible, and no minuteness can annihilate either. But material things, or magnitudes developed to the senses, are governed by laws which the existence of *space* or magnitude in the *abstract* does not involve. Such developed magnitudes, as for example, those composed of metals, of wood, water, earth, &c., do not always, and I think never, fill all the space within their boundaries. Their parts may be united by cohesion, but the lines which separate their parts, though infinitely diminished are not annihilated. And the bodies are filled with porosities which allow of the existence of other material elements of nature within them, as air, moisture, light, heat, electricity, &c. We know also, that all material or developed magnitudes are formed from *original elements* which had a separate atomic existence, and which have become united by their affinity, and hence they may be so divided that they *cannot be divided any further* without resolving themselves into their original elements, or when so divided they may perhaps, by forming new affinities, take some new form of existence, by which they may appear to be annihilated; and this we know is the process of nature, by which all material things are subjected to decay and renewal; but there is no such law governing space or magnitude in the abstract, which is subject to no decay or renewal. Therefore it is evident that in *material things* magnitudes are *not infinitely divisible* in the fullest sense of the term Infinity. I would, therefore, define the term infinity in its application to material things (of which alone we are cognizant) in a limited sense, and say that an infinity is one ultimate particle of whatever material or thing we are considering, such that it cannot be divided again without resolving itself into its original elements, and therefore such that it *cannot be less.*

This, I think, is the only comprehensible meaning which can be given to the term "infinity" in its relation to matter, and if we reason from things which are *incomprehensible*, we reason of things which we know nothing about, and must fall into error. If we apply the term infinity in its fullest sense to material things, it will result, that a drop of water may be divided just as many times as a square foot or an ocean of water, and we shall have one infinity greater than another infinity of the same thing, which is absurd. But if we apply the term as I have defined it, then no infinity *can exist* which is greater than another infinity of the same thing, and a most important truth is brought within comprehensible limits.

I have made these remarks concerning infinity, as being applicable to the propositions which follow, and to the *difference* between a line coinciding with the greatest limit of the area of any circle, and a line *inclosing* the same circle, which is infinity, in the sense in which I have defined it.

I shall here lay down as axioms certain truths which have been proved.

First. The circumference of a circle is a line outside of the circle thoroughly inclosing it, and of itself forms no part of the area of the circle. (Prop. v., chap. ii.)

Second. The line approximated by geometers, if it could be correctly determined, is a line coinciding with the greatest limit of the area of the circle, but not inclosing it. (Prop. I., Appendix.)

Third. The line approximated by geometers is consequently the circumference of a circle whose diameter is less than one in its relative value to the area of a circle. (Prop. I., III., and IV., chap. i., and prop. I., Appendix.)

Fourth. The difference between a line coinciding with the greatest limit of the area of any circle and a line inclosing the same circle, is an infinity, such that it cannot be less. (Prop. II., Appendix.)

Fifth. In material things an infinity equals one ultimate particle of whatever material or thing is under consideration, such that it cannot be less. (Prop. II., Appendix.)

Sixth. An infinity is a value, such that it is always capable of increase. (Prop. II., Appendix.)

I now propose to show, that by the method of geometers, the omission of the difference between the radius of a line *coinciding* with the greatest limit of the area of any circle and the radius of a line *inclosing* the same circle, being an infinity, the value of such infinity is increased in the process of bisection, so that it shall always equal one or more in the sixth decimal place at some great number of sides of a polygon; and may be increased, so that it shall equal circumference itself.

It will be seen that if the radius of the inscribed line, or line coinciding with the greatest limit of the area of the circle shall equal *finity*, then because the difference between the inscribed and circumscribed lines equals *infinity*, therefore the radius of the circumscribed line equals infinity added to finity. Therefore let finity—F, and let infinity added to finity —I F.*

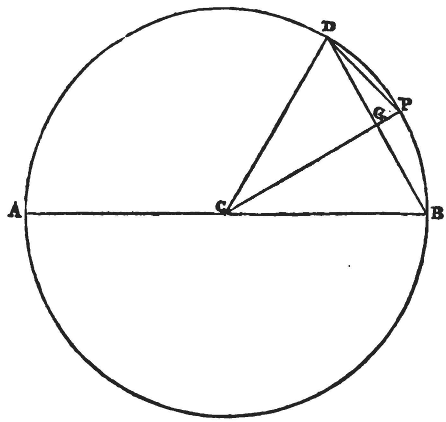
PROPOSITION III.

The value of that infinity which is the difference between the inscribed and circumscribed lines (axiom 4th), and which is omitted by geometers, is increased in the process of bisection of a circumference, so that at some great number of sides of a polygon it will always equal one or more in the sixth decimal place, and may be increased, until it shall equal circumference itself.

I now take Playfair's eighth proposition, first book, of the Supplement to the Elements of Geometry, in which he bisects a circumference to 6144 sides of a polygon.

* In examining this proposition, we cannot do better than to place a glass of water before us, and supposing the tumbler to be a perfect cylinder, let us then suppose the radius of the greatest possible line coinciding with the water =F, and the radius of the least possible line coinciding with the interior of the tumbler =I F, which last is the circumference of the circle.

PLATE XXX.



Let the radius C B (Plate XXX.) equal *one* of thirteen decimal places, that is, C B—1.0000000000000, then let C G equal the perpendicular on the chord of the arch of 6144 sides, then C G —.9999998692723 + then C B—C G—G P, and G P×A B—D P² and √D P²—D P, which is the chord of the arch of 12288 sides. Now let F (finity) equal radius —C B—1.0000000000000, then F—C G—G P, and G P is seen to equal 1307277— as follows:

F—1.0000000000000
C G— .9999998692723+
G P— 1307277—

Now let an infinity, such as I have defined it, equal one in the thirteenth decimal place, then I F (infinity added to finity) —1.0000000000001.—then I F—C G—G P and G P is seen to equal 1307278—as follows:

I F—1.0000000000001
C G— .9999998692723+
G P— 1307278—

It is seen that while radius (F and I F) are lines of *fourteen figures*, or a left hand unit and *thirteen* decimal places, G P is a line of only *seven figures*, or a left hand unit and *six* decimal places, and G P—I F—C G is seen to be greater by one in the *sixth* decimal place than G P—F—C G, and because G P × A B—D P^2 and √D P^2—D P, and D P×12288 (the number of sides of the polygon) gives the circumference, therefore G P—I F—C G will give a circumference greater in the *sixth decimal place* than G P—F—C G, viz., G P—F—C G×A B at 12288 sides of a polygon will give a circumference —3.1415925+, which is less than geometer's ratio, and G P—I F—G G × A B will give 3.1415948+which is *greater* than *my* ratio. It is evident, therefore, that if that infinity which is equal to the *essential difference* in the properties of straight lines and curved lines, and which is consequently equal to the *difference* between a line coinciding with the greatest limit of the area of any circle, and a line *inclosing* the same circle, shall equal one in the thirteenth decimal place of any line of figures, the omission of the value of that infinity, will, in the process of bisection, to 12288 sides of a polygon, be an error in the sixth decimal place of circumference.

Again, let F (finity) equal one of 16 decimal places or 1.0000000000000000, and let infinity equal one at the sixteenth decimal place, then I F (infinity added to finity) —1.0000000000000001. Now let the circumference be bisected fifteen times, this will give an inscribed polygon of 196608 sides, then let C G equal the perpendicular on the chord of the arch of 196608 sides of an inscribed polygon, then C G=.9999999998723363+ and F—C G is again seen to be a line of only seven figures, or a unit with six decimal places, and I F—C G is also again seen to be greater by one in the sixth decimal place than F—C G, and at 393216 sides of a polygon, an infinity which had a value of only one in the sixteenth decimal place, is increased in value in the process of bisection, so that it becomes one or more in the sixth decimal place of circumference. The error of geometers is here *palpable*—it does not even require the calculation to be made in order to demonstrate it,—the number of figures left when C G is deducted from F is alone sufficient to show at what point the perimeter of the polygon found, is less than the true circumference of the circle; and it is perfectly obvious, that this error arises from mechanical causes perceptible to our senses; and from an inherent property of numbers which cannot be obviated by any method of geometers. It is not a defect or discrepancy of numbers, but it is the perfection of their power, and is easily understood to have its origin in the essential difference in the properties of straight lines and curved lines. It is perfectly obvious also, from the examples given, that if we could go on with the process of bisection until the perpendicular on the chord of the arch should equal radius, the error arising from an infinity omitted at the first, would then equal circumference itself. It is equally obvious, that at whatever point in a line of figures we may place the value of infinity by hypothesis, whether at the sixteenth, twenty-fifth, fiftieth or even at 1000 decimal places from unit, by the process of bisection the value of such infinity will be increased, so that at some great number of sides it will equal one in the sixth decimal place, and finally equal circumference itself. I do not say that one in the sixteenth, twenty-fifth or any other particular decimal place *is an infinity;* but if a little estimate be made of its value, we may form some conception, whether it may, or may not, be an infinity, according to my definition of the term.

A difference which equals one in the sixteenth decimal place, is such, that if the magnitudes be miles, it is less than one hair's breadth in the distance from our earth to the sun! and it is less than the *four-thousandth* part of one hair's breadth in the circumference of our earth! Whether such a difference is equal to an infinity in the surface of a glass of water (to which I have requested reference in a note to this proposition; also in a note to proposition v., chapter ii.), or, in other words, equal to one particle of water in its least possible natural division, and hence equal to the difference between the *greatest* possible line coinciding with the water, and the *least* possible line coinciding with the interior of the tumbler inclosing the water, I leave to the decision of those whose perceptions and judgments are more acute than mine.

The principle shown in the examples given, is a clear one, and proves conclusively, that if the ocean were spread out in a circle, and the difference between a line *coinciding* with its utmost limit, and a line *inclosing* it, should be of the value of one particle of water in its least possible natural division, the omission of that value would, in the process of bisection by geometer's method, at some great number of sides of a polygon, become of the value of one or more in the sixth decimal place of circumference, and finally equal circumference itself.

The proposition is therefore demonstrated.

Having completely demonstrated in the six propositions of chapter i., that there is an essential difference in the properties of straight lines and curved lines, which has been entirely overlooked by geometers,—having proved, also, in the fifth proposition, chapter ii., that the circumference of a circle is a line *outside* of the circle, thoroughly inclosing it,—in the first proposition of this Appendix, that the line approximated by geometers is a line *coinciding* with the greatest limit of the area of the circle, but *not inclosing* it,—and by the last proposition, that the method of geometers is such, that by the difference in value between these two lines, they are liable to an error in the sixth decimal place (the point at which their ratio differs from mine): It would seem as if all these demonstrations were sufficiently conclusive, and that no professor of mathematics would, hereafter, object to my ratio of circumference and diameter, on the ground that "it differs from their approximation, in the sixth "decimal place," unless he can first disprove all these demonstrations, by some other means than by assuming to be true, just what has here been proved to be false, a very common way of repelling truth, and then dignifying it with the name of argument. But I have no idea that professors will so easily surrender the point. It is a part of human nature, that men who are joined to their idols will never let them go,—neither will *they*. As I have said in the introduction to this Appendix, "if "driven from one principle, they will fall back upon "another; and if beaten from all, they will return again "to the first;" and all we can do, is, to reduce them to this necessity, and there leave them.

In the last demonstration, I have introduced, in a note, for illustration, the natural lines which are seen in a glass of water, to show the difference between a line *coinciding* with, and a line *inclosing* the circle. It will be asserted, no doubt, as an objection to the truth shown in the last demonstration, that these two lines (the line of the water, and the line of the tumbler) meet at a line without breadth, which is the common boundary of both, and which they hence assume to be the common measure of both; in other words, that because the water and the tumbler are said to touch each other, that they are therefore equal. We must, therefore, anticipate their resort, and take away the subterfuge, before they turn to it. If they would make this objection, and abide by the principles it involves, it would be quite sufficient for my purpose, for thereby, they would admit that the circumference of the circle is a line *outside* of it, and it is then easily shown, that the difference between this line, and that which they measure, is that infinity for which I contend, and the omission of its value lays them liable to an error in the sixth decimal place. But they will not abide in argument, even by the principles of their own objections, because it is not their purpose to find the truth, but only to object, lest in finding the truth, *they* should be found in error. I shall, therefore, treat the subject on its true merits, and show that no curved line, or line of circumference, can be, at the same time, the common boundary, and the common measure, of its two sides.

Before proceeding to demonstrate this principle, I shall here again lay down as axioms, certain truths, which have been proved, or are self-evident.

First. Space is infinitely divisible. (Proposition II., Appendix.)

Second. Any imaginary line (not a material line), which shall have breadth, is equal to the same portion of space.

Third. Any such imaginary line is, therefore, infinitely divisible.

Fourth. Any such imaginary line may, therefore, be divided, until each part or division is less than any magnitude which is, or can be, developed to our senses.*

* Theoretic men have a science, which they call the "science of vanishing quantities," and which, I think, has some affinity to this idea of the existence of magnitude in the abstract, beyond the means of development by any magnifying power. I know nothing about it,—I only know that they have such a science, by which they assume to prove that a thing may be annihilated, and yet continue to exist. I can well understand how abstract space or magnitude may be diminished, until it is entirely beyond our perceptions by any aid which we can control. And I can well understand how a developed magnitude may, by some process of nature, become so divided, as to return to its original elements; or, by forming some new affinities, pass to some new form of existence, and be seemingly destroyed. But I cannot understand how a thing can exist, and yet not exist at the same time. The thought appears to me to have its origin in the same class of abstract absurdities, which calls an infinity "nothing," and then, as occasion may require, seeks to prove that nothing equals something. If asked my opinion, what is the limit of developed magnitudes, I would answer, that, so far as I am able to reason, without much reflection on the subject, the atmosphere we breathe, seems to be the boundary line. Being

Fifth. At whatever point the division of such a line may be arrested, because the sum of all the parts is equal to the whole; therefore, each part must have breadth, though the breadth of each part may be such, that no conceivable number of them would form a developed magnitude.

Sixth. One line cannot occupy two places at the same time; neither can two lines be in one and the same place, at the same time.

Seventh. Two lines without breadth, cannot exist with no breadth between them.

Eighth. The existence of shape signifies limit; hence, no shape can exist without a boundary line definitely located, which forms no part of the shape itself, which boundary is its circumference.

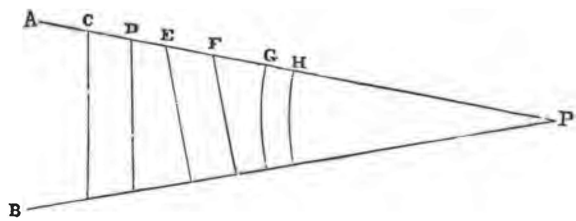
PROPOSITION IV.

No two lines lying in the same plane, parallel to each other, and between two other straight lines, which are at

the medium of light which discloses magnitude, it cannot, itself, be developed to the senses. We can feel its mass, by its own motion, or its resistance to motion; but we can neither see, nor feel, its separate particles, though convinced of its existence in that form, by the evidence of its motion. I should infer, from these facts, that particles less than those of the atmosphere, cannot be made perceptible to our senses; and that particles greater than those of the atmosphere, are within the scope of possible development.

an angle to each other, can possibly coincide, and be equal, except they shall become one and the same line.

PLATE XXXI.

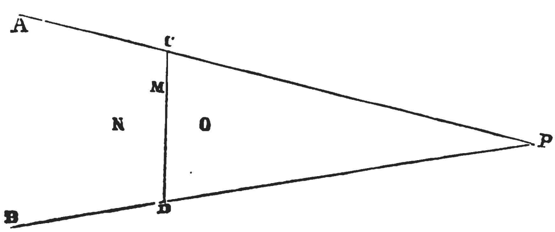


Now, let A and B (Plate XXXI.) be two straight lines, at an angle to each other. They are seen to meet at the point P, therefore, P may be supposed to be the center of a circle. Now, let C and D be two other lines parallel to each other, lying in the same plane, between A and B, and at any angle to A or B. It is seen that C and D are not equal; that which is farthest from the center, P (C), being greatest, and that which is nearest the center, P (D), being least; and because one is greater than the other, therefore, if brought together, they cannot *coincide*, but *in part;* and if either C or D be divided through the center, lengthwise, in halves, the halves cannot coincide; and if they be divided thus, to infinity, no one part of either C or D can coincide with any other part, because no one part of such division is equal to any other part,—that which is farthest from the center P being greatest, and that which is nearest the center P being least. So also with E and F, or G and H, and if either be divided to infinity, none of the parts so divided can coincide with any other part, because no one part of such division is equal to any other part, but each part coincides with, and is equal to itself *only*. *The proposition is therefore demonstrated.*

PROPOSITION V.

All lines which have a fixed and definite locality, must have breadth, whether they be lines of circumference, or lines of division.

PLATE XXXII.



Let A and B (Plate XXXII.) be two straight lines at an angle to each other as in the fourth proposition, and meeting at the point P, which is supposed to be the center of a circle. Let A and B form two sides of a developed magnitude equal to M, and let N and O be two divisions of that magnitude. Now let an imaginary line supposed to be without breadth fall on M dividing it into two parts as at C D. It is known that every part of M which is nearer to the center P than such imaginary line, will fall on the inside (O), and every part of M which is farther from the center P than such imaginary line will fall on the outside (N), therefore (proposition IV.), every part of N parallel to C D is greater than any part of O parallel to C D. And because every part of N is farther from the center P than any part of O, therefore the line C D being a line of division has breadth, however we may have imagined it, though such breadth may be a less portion of space than any separate existing portion of the material or thing divided. *The proposition is therefore demonstrated.*

From the demonstration of the last two propositions and their preceding axioms these necessary deductions follow:

First. In the illustration of the glass of water, the water and the tumbler being two materials which cannot mingle, they therefore occupy two places, and (axiom 6th) "one line cannot occupy two places at the same "time," therefore the lines conciding with each (the water and the tumbler) are not one and the same line. And (axiom 7th) "two lines without breadth cannot "exist with no breadth between them," therefore the line between the water and the tumbler *has breadth*, though such breadth is evidently less than one particle of water in its least possible natural division (see Note to prop. v., chap. ii.).

Secondly. *Every* part of the tumbler is farther from the center than *any* part of the water, hence (prop. IV.) the line of the tumbler is greater than the line of the water, therefore a line between them being greater than the inside and less than the outside, *cannot* be the *common measure* of both its sides,—it must be assumed as the measure of one or the other of its sides, but not of both, and in this particular it differs essentially from a straight line. It is the common boundary of both its sides only, as it limits *extension from* the center outward, and diminution *towards the center* inward, which are opposite qualities. From the mere fact of its being the common boundary, it cannot therefore be the *common measure of extension* on both its sides.

Third. The line between the water and the tumbler being a line of division which separates all parts of the water from all parts of the tumbler, leaving one wholly inside and the other wholly outside, therefore (proposition v.) that line has breadth, and being defined and considered as having breadth, therefore when we say that the circumference of the water is the least possible line of the tumbler, we mean of course, that line which limits the *diminution* of the interior of the tumbler *toward* the center, and which limits the *extension* of the surface of the water *from* the center, and is the common *boundary* of both, but not the common measure of *extension* in both. But if the line between the water and the tumbler be defined and considered as having *no breadth*, then the line which would limit the extension of the water would be a line *coinciding* with the tumbler and not interior to it, and the line which would limit the diminution of the tumbler toward the center would be a line coinciding with the water and not outside of it. And because the circumference of a circle is a line outside of the circle thoroughly inclosing it (prop. v., chap. ii.), therefore if the definition of a line shall be that it has *no breadth*, then the circumference of the water is a line coinciding with a part of the tumbler. But if the definition of a line shall be such that all lines which are lines of division and have a fixed and defined locality *have breadth*, though such breadth may be infinitely diminished (axioms 3th, 4th, and 5th), then the line between the water and the tumbler, having breadth, and being wholly outside of the water, is the circumference of the circle, and it is evident (axiom 8th) that no circle can exist in nature which has not a fixed boundary or line of circumference which separates it from all surrounding things. These differences which grow out of the definition of a line require to be carefully considered.

Fourth. It is self-evident that *contact* is not *union*, and it is a mathematical error to consider them as *one*. Therefore if two magnitudes be separated by any given line or space between them, and if they are then brought together until they are in contact or seem to touch each other, the line or space between them is only divided or diminished—it is not annihilated. And because space is infinitely divisible (axiom 1st and prop. II.), therefore it may be infinitely diminished, but it can never be annihilated unless the two sides shall become one, which is impossible. They may approach so closely as to exclude all other matter and cohesion may take place by affinity of the parts, but the lines of contact or cohesion are still in existence, terminating the boundaries of the two sides, and hence there is space between them, and if not, then space is *not infinitely* divisible. Hence the water and the tumbler being wholly distinct and unmingled there is space or breadth of line between them, though such breadth is evidently diminished until it is less than one particle of water in its least possible natural division.

The foregoing demonstrations and the deductions from them, naturally lead to some few remarks respecting the properties of numbers, and of magnitudes, with the mathematical definitions of a line, and a point, and their application in geometry.

I will not undertake to say, that because it is proved that all lines having a fixed and definite locality have breadth also, that under the circumstances, it is absolutely necessary in *all cases* to change the mathematical definition of a line or a point; but I do say, that to regard these definitions, in their abstract, arbitrary, and limited signification as essential to truth, is *wholly unnecessary*.

Nature will work in her own way in spite of our definitions, and if in geometry *we* work according to nature's truth, the result will be correct, whether we define lines as having breadth or no breadth.

The science of mathematics in the most universal sense in which the term can be applied, is, I think, the science of numbers, which have infinite capacity, both as to notation and enumeration.

Algebra means *nothing*, or it may mean either one thing or another, as suits the fancy of him who works in it, until its results are reduced to numbers; and then there is no value in the universe, either simple, compound, or relative, which some notation, enumeration, or relative fraction of numbers is not capable of expressing. A little examination into the character of numbers, will, I think, be sufficient to convince us of this truth. And what are numbers? Have we any definite ideas of their nature, capacities, origin, or end? Are they a creation of God, existing by his power independent of other things? or are they simply the result of a previous necessity which the order of creation only fulfills, and which hence follow as a mere consequence of the perceptions with which the Creator has endowed our minds? I am compelled to think the latter, and hence whether in the material or immaterial world, to our perceptions, numbers and existences are identical and inseparable. Before creation began then, numbers had no existence, except in the infinite eternal ONE. But when the first particle of original matter was brought into being it constituted the *finite one*, and the second particle *two*, and finite and progressive numbers then had their beginning, and have ever since, with every succeeding production, been moving forward towards the INFINITE. But our notation of decimal numbers, is only one of the forms which nature employs for herself; and it is because they *are* one of the forms which nature employs for herself, which gives them their power, accuracy and clearness; hence decimal numbers will be found to have an important part in the order of created things. But nature also employs other forms or notations of numbers besides decimals, and whenever she does so, decimal numbers are not accurate, or rather they have not the power to tell us the exact truth without the loss of some fraction or remainder.

We know very well that all notations of numbers must have their beginning in *one*, which is less, and their end in *one*, which is greater; thus when we say "one "thousand," we simply mean a thousand ones, or if we consider it in its unity, we mean *one*, a thousand times greater than the original one. Hence it is plainly seen that when the first original particles of matter were collected together and moulded into shape,—when our earth assumed her opaque body, and when the first star sprang into existence, it was nothing else but a following out of the progression of numbers from an original creation of *one* less, to the accumulation of *one* greater. At least such is the only understanding which our perceptions can give us of the order of production in created things: and it appears to me, that in material things, *truth* is nothing more than the perfect agreement between nature and our perceptions, and error is their disagreement. Hence if we had been differently endowed, natural truth might have been to us quite another thing, and to our minds numbers might not have been what they now are.

If then this agreement between the order of production in nature and our natural perceptions *be truth*, then it is self-evident, that there exists not in the wide universe, a single particle of matter, or combination of particles into shapes or things, which numbers have not told out with unerring exactness; and what numbers ace, they can do again. It is evident, therefore, that in the material world, numbers and things, that is, shapes and magnitudes, are identical and inseparable, and we cannot comprehend one without the other, therefore, *numbers* are the legitimate medium of determining mathematical truth, and no solution can be complete until it is reduced to numbers. It is an error, therefore, of mathematicians, to give to numbers the inferior place in mathematical science.

The science of geometry, as a part of mathematics, is generally defined as the science of measure, or of quantity; but I think this definition is too limited, and that it may, with greater justice, be termed the science of perfect mechanics, by which all forms and proportions are *produced*, as well as measured, and their relative properties and values determined. If, for example, we would make any shape or form out of brass or other material, in order that its proportions may be as nearly accurate as possible, we must first *produce it* geometrically, and then make one from brass, as near like the geometrical form produced, as we can. The first is, then, in principle, a *perfect* form, according to nature's working; but the second is only an imitation, and is imperfect, by reason of our inability to measure or detect so small a quantity of matter as one original particle, which is the only perfect standard of measure; and if the form of brass shall differ from the true geometrical form but one particle, it is imperfect. The first is an operation of geometry, and is nature's work; the second is a mechanical operation, and is the work of the artisan. The first is the development of the pure principles which govern the form; the second is only the labor of the workman with his tools. If, therefore, we separate geometry from mechanics, we leave the latter without a science, and degrade the mechanic arts to the character of a blind imitation, without rule or principle. The really scientific mechanic is one who constructs his work according to geometrical principles, and the only difference between his work and his principles is, that his principles are perfect, being according to nature's laws, but his work is imperfect, from lack of skill to make it more perfect. The power of mechanics is altogether constructive, it is not creative; she can fashion things, but she cannot make them,—we must first furnish her with materials, and she will then mould them; and being furnished with the necessary parts of things, she can put them together. So, also, with geometry; she can create nothing,—all her powers are constructive, only; she finds all her proportions in magnitudes and forms, and we have seen that numbers and magnitudes, or forms, are inseparable. These, then (numbers and magnitudes), are the materials of geometry, and until she is furnished with these, she can do nothing. Let it be required of geometry to produce a hexagon, and she requires you to furnish her with six equilateral triangles, and placing them together, shows you a hexagon; and in producing this hexagon, geometry has done no more than mechanically to construct a form out of other forms, and build up a magnitude out of lesser magnitudes; and this is the limit of her power, because she can create nothing. If you now require to know the value of this hexagon, in proportion to other shapes, decimal numbers will tell you that each side of the hexagon being one, its value in proportion to the equilateral triangle of an equal side is *six*, and in proportion to the square, it is the square root of 6.75. If, in this last, you wish for any more definite expression than square root, decimal numbers will tell you to go and acquaint yourself with some other notation, besides decimals, which can give the needed fraction, or otherwise to sit down in ignorance. Now, let it be required of geometry to produce a form without the use of magnitudes, and she tells you that she can do no such thing, that form and magnitude are inseparable ideas,—that all shapes and forms have both extension and limit, and are, therefore, finite,—that she deals in finite magnitudes, and nothing else,—that abstract space is *infinite*, and if she ever considers space, she considers it only in finite portions, and in reference to some developed magnitude having the same boundaries; hence, if geometry be required to measure any portion of space, she demands to be furnished with some magnitude as a standard of measure, or otherwise she cannot do it.

Let it be required of geometry to give a fixed and definite locality to a line without breadth, or to a point in space, without magnitude, and she will tell you that locality, place, and position, are all of the same import, and all of them mean a *portion of space*,—that there can be no division of space equal nothing (see page 158); and, therefore, there can be no locality, place or position, without magnitude; and hence, your line without breadth, and position without magnitude, are fallacies, both in nature and mechanics, and therefore, beyond her power.

Ask geometry to *measure a form* by a line without breadth, and if you please, let the form be a hexagon of uncertain size. Geometry at once answers you, that a line without breadth has no existence, and if you furnish her with no other materials she cannot do it,—that magnitude is only magnitude by comparison with some standard of measure, and hence things can only be measured by comparison with other things, and therefore, it is out of her power to measure something, by comparing it with nothing. Even a standard of measure (an artificial standard) is without meaning, only, as it refers to some other standard; as, for example, a carpenter's foot-rule is without expression as to quantity, only as we mentally refer the rule to the foot. When we measure a thing, therefore, we do nothing but simply determine its quantity, in its relation to some other quantity already determined; how, then, can relative quantity be determined, by comparing any quantity with itself, or by comparing it with no quantity? But furnish geometry with a positive magnitude, which shall be her standard of measure, and which she can apply to any part of the hexagon, and then knowing what the form is, numbers will perform the rest, and you may imagine your line to be just what you please—as having breadth, or no breadth, it is all the same to geometry. Having determined the extension of the form in one direction, by a positive magnitude, or line with breadth, and knowing what the form is, geometry can now determine its extension in all directions, from any point, and the result will be correct, and geometry is satisfied; but without the help of this positive magnitude, you could never have known anything of the value of the hexagon.

And now if there be anything in the world to personate common sense, I would ask, has geometry measured this hexagon by a line with breadth or by one without breadth? Certainly by one *with breadth*, will be the answer, and by no other means did geometry ever yet measure anything? It seems therefore that geometry can do nothing with a line *without* breadth independently and alone,—you must first furnish her with a line *with breadth* with which to measure the imaginary one *without breadth*, before she can proceed one step in her work towards determining the quantity of any figure. In proposition v. (this Appendix) I have shown that in dividing any material or developed magnitude, geometry does it by a line *with breadth*, and it has been shown also, that such a line may exist, and yet be less than any portion of the material magnitude divided. Let us now see what sort of a line geometers do actually use in dividing a magnitude. Let the magnitude to be divided be a plate of gold one inch square, and to be divided equally. We know that gold is constituted of original particles, because it can be dissolved or dispersed and its particles collected together again and deposited in a new place, the particles cohering as before; therefore let the plate to be divided be of the imaginary thickness of one original particle of gold. The principles of geometry fix the line *where* it is to be divided with perfect accuracy, so that just as many particles shall lie on one side of the line as on the other. But the geometer not being able to locate, to perceive, or to understand the exact place of his imaginary line *without breadth*, in order to aid this deficiency of his perceptions, he draws a line *with breadth* across the face of the plate of gold at the points indicated by geometry; and knowing that this ine by its breadth covers a portion of the gold, he proceeds mentally to diminish the line towards the center, until its breadth is *less* than any portion of the gold, or in other words, until its breadth is equal to the lines of cohesion which unite the particles of gold. And the magnitude or plate of gold is thus mentally divided, without the loss of any portion of its quantity; and because the line of contact or cohesion of the particles of gold has two sides, leaving all the particles of one half on one side of it, and all the particles of the other half on the other side, it therefore has breadth, though such breadth is diminished, until it is less than one particle of gold, and less than any of our perceptions of quantity. The particles of gold do not, by their cohesion, unite, to become one and the same,—they simply approach each other within a distance, such as to exclude all grosser matter, while each particle of gold remains as before distinct from the other.

And now I think that no geometer who is capable of examining the nature and extent of his own perceptions, will tell me, that in dividing any magnitude geometrically, he has ever made use of a line mentally or physically, in any other way than just as I have described above; and if so, then no geometer has ever yet (except in name) used a line without breadth in dividing any magnitude. As an evidence of the truth of this conclusion, we need only make the attempt, mentally, to fix the position of a point without magnitude, or the locality of a line without breadth; as we diminish toward the center, our very thought expires with the expiring magnitude, and we have neither recollection nor comprehension of its exact place.

I have, in another place, explained natural truth to be nothing more than the agreement between nature and our perceptions, and error as their disagreement. Now it appears to me that in the application of lines and magnitudes to geometry, the explanation which I have given of their use, forms a perfect agreement between the operations of nature, and the perceptions with which nature has endowed us; and if so, then it is true; and the ideas of lines without breadth, and position without magnitude are misnomers,—mere illusions of the imagination,—altogether unnecessary,—wholly without use, and in opposition to all natural truth and evidence.

The mathematical definition of a line,—"that which "has length but not breadth," at first strikes the mind as an absurdity, because it implies quantity of one kind, and yet it has no existence. But on examining it with the application made of it by geometers, it is found to mean *mere distance* from one point to another, and where nothing but mere distance is intended, it is wholly immaterial to the result whether the line shall have breadth or not. We may consider it an inch wide, or a foot wide, or as broad as it is long, and no difference will follow in the result. The definition is therefore exceedingly imperfect, even if true. But it is *not true* in nature, and therefore *not true* in geometry, because nature is the perfection of geometry. Lines in their practical application in nature, and consequently in geometry, have a more enlarged use and meaning than mere distance. They in reality constitute the divisions of all magnitudes, and the boundaries of all shapes,—offices which mere distance is incapable of performing. We have seen that shape or form (which is identical with magnitude), is essential to the first principles of geometry. We have seen also that no shape can have any positive existence without *limit*, and a boundary clearly defined, definitely located, and separating it from all surrounding things. And the proposition is self-evident that lines which perform these offices *must have a positive existence*, and therefore *must have breadth*,—therefore the circumference of a circle *must have breadth.*

A definition does not necessarily form any part of mathematical truth. It is only a part of the method by which truth is determined, and is not always essential even to this; as we have already seen, that in certain circumstances, it is entirely immaterial to the result what the definition may be. The chief object of a definition is to enable the mind to comprehend a truth when it is determined, rather than to constitute any part of the truth itself. A definition must be conformable to truth in the circumstances in which it is used. Hence the definition of a line without breadth, being conformable to truth in such circumstances, is a good definition in all those cases where distance only is intended, in which it is entirely immaterial what may be the breadth of a line or whether it has any at all. But in all those cases in which a line *with* breadth differs in value from one without breadth (which it always does in a continuous line of circumference), the mathematical definition is *not conformable* to the truth of nature, and therefore leads to error.

I do not say, therefore, that it is absolutely necessary *in all cases*, to change the mathematical definition of a line, but I do say that it is wholly unnecessary to consider the definition as a mathematical truth, which is fixed and unalterable, and that *it is* absolutely necessary to consider the circumstances in which a line is used, and to modify or change the definition in all those cases in which a line with breadth differs in value from one without breadth, as in the circumference of a circle. In such cases, I would define a line as that which has length and the least possible breadth with locality, and I would define a point as that which has position and the least possible magnitude with locality.

It will be seen, however, by all who examine the whole course of my reasoning attentively, that lines having breadth, is *not a condition* on which the Quadrature of the Circle is based, but only a *conclusion* to which the developments as they proceed inevitably lead us. The twelve propositions of chapter ii. which embrace the whole demonstration of the Quadrature, rest entirely on the relative properties of shapes, and of straight lines to curved lines, and *are wholly independent of the fact* whether lines have breadth or not. If, therefore, any one should choose for any reason to reject the idea of lines having breadth and points having magnitude, it cannot in any wise affect the truth of these demonstrations or the main object of this publication.

We have seen, in the course of the foregoing reasoning, that *shape* or *form* is essential, both to the *principles* of geometry, and to the *practice* of geometry, and that form and magnitude are identical and inseparable ideas. We have seen, also, that *numbers* and *things* (which, in the material world, are the same as magnitudes) are also inseparable. We have seen, also, that geometry, being the science of *quantity*, can only compare indefinite quantity with other quantities which are known, and definite. We have seen, also, that geometry—being, as she is, the basis of pure mechanics—when she is required to *produce* a *form*, she does it by using, or putting together, *other forms*, just as we build a ship, or a house; with this difference, only, that all her forms are perfect, but ours, from lack of skill, are imperfect. And when geometry is required to produce a magnitude, she does it with the use of lesser magnitudes, which are her standards of measure, and without the use of both form and magnitude, it is not in the power of geometry, either to produce anything, or to measure anything; she positively refuses to proceed one step, either in the development, or measurement, of any form, until she is furnished with a positive magnitude, as a standard of measure, which she can consider, and treat, as *one;* and no abstract solution of any geometrical problem has any meaning in it, until the result is compared with some known quantity of definite form, as a standard of measure. We have seen, also, that the power of geometry is limited to finite things, or things having *limit* or *boundary*,—that she considers space, only by comparing it with finite and definite magnitudes; that *infinite space*, or *infinite magnitude*, is entirely beyond her reach or comprehension; and that all abstractions are necessarily infinities.

These conclusions form a perfect agreement between the operations or developments of nature, such as we can understand, and the perceptions and powers of comprehension, which nature has given *us*, and *they* are, *therefore, true*. What then becomes of the idea of the existence of geometry as an abstract and metaphysical science? In my opinion, the notion of abstractions in geometry sinks at once to a level in value with the chimeras of a sick man's brain; they are mere illusions, floating in the mind of the operator, without identifying themselves with the practical truths of which they are only the mental images, and, I believe, were never thought of by geometers, until the introduction of algebra, and its abstract mode of reasoning, which is without use or meaning, until it is brought down to the standard of some definite form and magnitude. I cannot, therefore, do otherwise, than conclude, that geometry has to do with *nothing but the relations of physical or material things*, and is, therefore, purely *a physical science.*

And since all that is known in geometry includes but a very small portion of either the general principles, or individual truths, which govern the relations of things to one another, it is self-evident, that the science, as practiced, is open to improvement, and capable of progress, just as much as the mechanic arts, the study of chemistry, or anything else; and notwithstanding its many conveniences, I regard the use of algebra in geometrical demonstration, as deserving no higher character than that of an ingenious invention to supply the lack of a knowledge of numbers; and I regard the consideration of geometry as an *abstract* and *metaphysical* science, as a sort of *ignis fatuus* light, which, by blinding the eyes of the beholder, renders more obscure, everything else around him, and which, from its first introduction, until now, has been the chief bar to a rapid progress in the science, which may be realized the moment this idea of abstractions is dismissed from our reason.

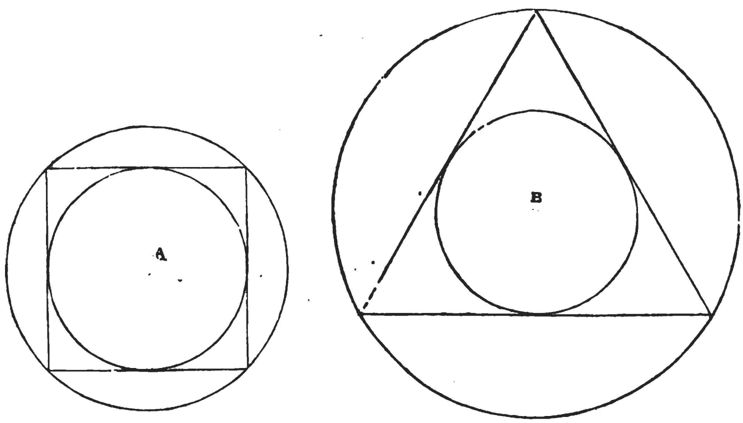
The opposite duplicate ratio of the equilateral triangle and the circle.

Duplicate ratio is a universal property of area, hence, the square being the standard of value, therefore, the square of diameter is duplicate ratio, and all superficial magnitudes, of any shape, are to each other of the same shape, in ratio of area, as the squares of their diameters, or in *duplicate* ratio. Explained in its simplest and most practical form, duplicate ratio of area means only, that the increase of area in any shape, is in duplicate ratio to the increase of circumference and diameter; that is to say, that area quadruples as often as circumference and diameter double; hence, because the equilateral triangle and the circle are, in their relations to the square, in the *opposite* duplicate ratio to one another (propositions VIII. and IX., chapter ii); therefore—

PROPOSITION VI.

The circle inscribed and circumscribed about an equilateral triangle, is in duplicate ratio to the circle inscribed and circumscribed about a square.

PLATE XXXIII.



Let the areas of the inscribed circles A and B equal one another; then the diameters of A and B are also equal. Now, let the side of the square circumscribing A equal one, then the diameter of A = 1, and because the diameter of the circumscribed circle equals the diagonal of the square, therefore, the diameter of the circumscribed circle = $\sqrt{2}$, and because the areas of all shapes are to others of the same shape, in duplicate ratio, or as the squares of their diameters, therefore, the area of the circle circumscribed about the square, twice equals the area of the circle *inscribed* in the same square. Now, the diameters of A and B are equal to one another; hence, the diameter of B — 1. It is proved by geometry, that the perpendicular of the triangle circumscribing B equals the diameter of B, plus the radius of B; therefore, the radius of the circle circumscribing the triangle, twice equals the radius of B, and 2 × 2 — 4; therefore, the area of the circle circumscribed about the triangle, four times equals the area of the circle *inscribed* in the same triangle. *The proposition is therefore demonstrated.*

It is evident from the foregoing proposition and its demonstration, that the equilateral triangle and the circle are, in their relations to the square, in some form or other, in duplicate ratio to one another. In what form this duplicate ratio exists, remains to be proved, if not already sufficiently proved in prop. VIII. and IX., chap. ii. In proceeding to prove this, I will first state certain truths, which are self-evident, or have been definitely proved already.

First. Circumference and radius (and not the square of diameter) are the only natural and legitimate elements of area by which all regular shapes may be measured alike and made equal to one another. (Prop. VII., chap. ii.)

Second. The equilateral triangle and the circle are exactly opposite to one another in the elements of their construction, which are circumference and radius. (Prop. VIII. and IX., chap. ii.)

Third. The equilateral triangle is the primary of all shapes in nature formed of straight lines and of equal sides and angles (prop. VIII., chap. ii.), and has the least number of sides of any shape in nature formed of straight lines, and the circle is the ultimatum of nature in the extension of the number of sides. Therefore,

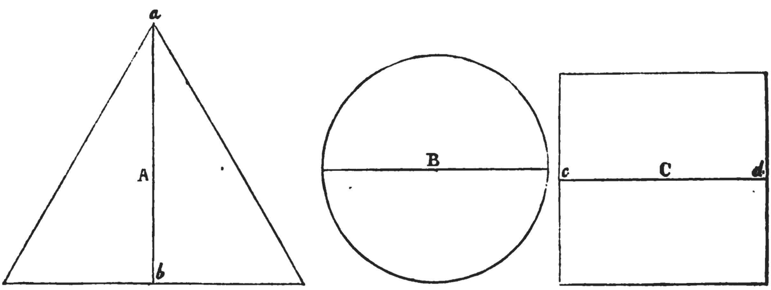
PROPOSITION VII.

In all *the elements of their construction which serve to increase or diminish area, the equilateral triangle and the circle are exactly* opposite *to one another in respect to the* greatest *and the* least *of any shapes in nature, and hence they are opposite to one another in ratio of the squares of their diameters, or in duplicate ratio.*

Now it is one of the plainest principles of geometry and arithmetic, that if G be greatest and L be least, then G×L and L×G are equal, because they are reciprocals. Hence it would appear on general principles, that the circle and the equilateral triangle should be equal, because one has the *greatest* possible radius and the *least* possible circumference of any regular shape in nature, and the other has the *least* possible radius and the greatest possible circumference of any regular shape, and half the circumference multiplied by radius are the only legitimate elements of area in each, by which they may be measured alike. But although this is a general principle in regard to *reciprocals in numbers*, yet it is only true in geometry when G and L are of corresponding or equal proportionate values. But in the relations of circumference and radius in respect to their relative values in area, *radius* is greatest, and *circumference* is least (radius=1, circumference =4), and because in the reversal of the order of *greatest* and *least* in the circle in its relation to the triangle, the greatest in relative value (radius) is also made greatest in relative magnitude, therefore the circle and the triangle are *not equal* as reciprocals of corresponding value are equal, but *opposites* in ratio, and the circle is to the triangle in its relation to the square as G×G, and the triangle is to the circle in its relation to the square as L÷L, because G×G and L÷L are in opposite ratio to G×L or L×G, and hence the square, or square of diameter, being made the artificial basis of area, they are opposite to one another in ratio of the squares of their diameters, or in the proportion of square and square root.

The term opposite signifies an intermediate, or a point, relative to which, the things spoken of are opposite to one another, and in this case the thing necessary to be known is, the point in numbers relative to which the circle and the triangle are opposite to one another in the proportion of the square and square root. Therefore let A be an equilateral triangle, B a circle, and C a square, each equal *one* in area.

PLATE XXXIV.



Now because A is a shape formed of straight lines and angles, therefore the value of *a b* may be known, but because B is formed of curved lines, therefore (by hypothesis) none of the parts of B are known, hence the point in numbers relatively to which the two shapes are opposite to one another must be determined from A alone. It has been demonstrated (prop. IX., chap. ii.) that if C and B are equal in area, then the diameter of C (*c d*) in its fractional relation to B is in the opposite duplicate ratio to *a b*, and the area of C ($c\ d^2$) in its fractional relation to B, is in like opposite duplicate ratio to $a\ b^2$.

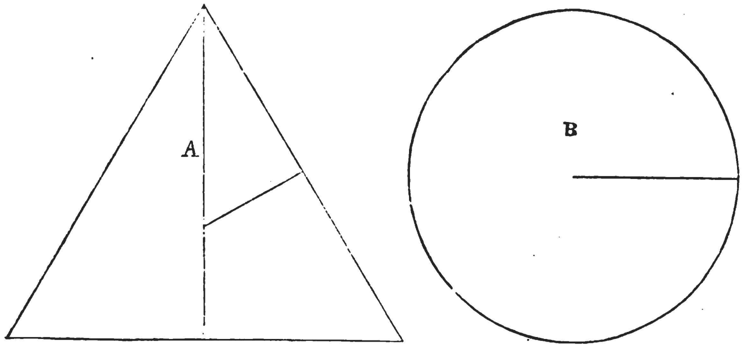
Now when the area of A—1, then *a b* is found to equal 1.316074+ and *a b* expressed in decimal figures is √

seen to be an infinite fraction, therefore, *a b* is an imperfect number, and all our ideas of fractions or magnitudes in order to be definite must be formed from whole and perfect numbers or units; for we cannot conceive of a fraction unless it shall have reference to some unit: therefore square *a b*, and $a\ b^2$ is seen to equal 1.7320508+ which is still an imperfect number, then square it again, and $a\ b^2 \times a\ b^2 = 3$, which is a whole and perfect number; therefore 3 is the first point in numbers from which the opposite ratio to *a b* may be deduced in *whole and perfect numbers.* Now because $a\ b = \sqrt{\sqrt{3}}$, therefore *c d*, being in the opposite duplicate ratio, $=3^2 \times 3^2$ viz., $3 \times 3 = 9 \times 9 = 81$, and 81 is seen to be the smallest number which can be found which is in the opposite duplicate ratio to *a b*, when *a b* is brought to a whole and perfect number; hence $81 \times 81 = 6561$, is the smallest whole and perfect number by which the fractional area of the circle and the square are equal to one another when the whole area equals one. (Prop. x. and xi., chap. ii.) The number 3 is therefore the point in numbers relatively to which, the circle and the triangle, in their fractional relations to the square are opposite to one another in duplicate ratio, or in the proportion of the square and square root.

The *opposite* ratio is simply a *necessity* resulting from a universal law of nature. The planets could not move forward in their courses without the action of opposite forces (centrifugal, centripetal). The north pole could not exist without an opposite or south pole. A shape or figure of one side cannot exist without another and opposite side, and no point can be fixed within the boundaries of nature which is not capable of opposite extension to the utmost limits of nature. In fine, no intermediate can possibly exist without its opposites. Hence the square being an intermediate shape and being made the standard of value, its opposites are the two extremes of nature in the production of shapes. And if we examine the whole subject in accordance with this universal law, it will be seen that in the production of shapes, the opposite duplicate ratio is a pre-existing necesssity. It is self-evident that shapes, which are thus opposite to one another in ratio, must, in their relative construction to the square, form the two extremes of nature in respect to all their elements which serve to create, increase, or diminish area, and hence, that not more than two shapes *can exist* at the same time which are thus opposite one another. The whole course of development in this work shows conclusively that the equilateral triangle and the circle are the *only two shapes* in *existence* possessing the qualities necessary to render them opposite to one another in duplicate ratio, or as the squares of their diameters. The truth of their relation to one another in their relative value to the square by opposite duplicate ratio is therefore self-evident. The square being made *one*, it is an intermediate shape relatively to which the two extremes in nature are necessarily opposite to one another. Any other conclusion would, I believe, in the nature of things be an absurdity.

The reciprocals of numbers by which the parts of the circle and the triangle may be made numerically to equal one, are to be found by reversing the order of the parts of each; therefore, if we multiply the greatest in relative value, by the greatest in relative magnitude, and the least in relative value, by the least in relative magnitude, then the two products multiplied together, equal one.

PLATE XXXV.



Therefore, if the areas of A and B each equal one, then, if the radius of B, multiplied by half the circumference of A, shall equal C, and the radius of A, multiplied by half the circumference B, shall equal D, then C × D = 1. Also, if the radius of A be divided by half the circumference of B, and the radius of B be divided by half the circumference of A, the two products will be equal to one another; but if the area of A = 1, and the diameter of B equal 1, then C × D equals the area of B.

This reciprocity of parts, and hence of numbers also, exists between the circle and all regular shapes whatsoever, which are formed of straight lines, when the area of each equals one. But all other shapes, *except the triangle*, lack the condition of being opposite to the circle, *in the elements of their construction*, in the particulars of the greatest and the least which are possible in nature; hence, *no other* shape, except the circle and the equilateral triangle, *can be* in the opposite duplicate ratio to one another, but *all other* regular shapes, formed of straight lines, and of any greater number of sides than the square, *approach* to the opposite ratio, "*ad infinitum*," in proportion to the extension of the number of sides.

It has been suggested by some friends, that I should add some examples of my method of deducing arithmetical truths from the problem of the circle, but this would extend a work already larger than I intended, and until my principles of reasoning, and the truth of my ratio of circumference and diameter shall be acknowledged, would be productive of no other benefit than the gratification of curiosity.

REVIEW OF MR. SMITH'S NEW ELEMENTS OF GEOMETRY.

Since the foregoing work was written, and with slight emendation, was prepared for the press in the form in which it now appears, a work has been published by Mr. Seba Smith, entitled "*New Elements of Geometry*," which has excited some attention and some discussion.

As Mr. Smith remarks in his preface to his work, his attention to the subject of Geometry was first excited by his examination of my demonstration of the Quadrature. About four years since Mr. Smith was, for a year or more, a close student of my papers, of which this work is only a portion in a condensed and regulated form. In his "New Elements" Mr. Smith has simply taken the general principles which I have developed, and applied them as he no doubt believes, to making further discoveries in geometry ; and in some particulars his success is beyond a doubt.

The chief principles he has made use of different from the commonly received Elements, which are contained in this work, and which were all fully developed and set forth in my original papers of which he was the close student, are these :—*First,* All definitely located lines necessarily have breadth. *Secondly,* Lines and areas are therefore equal to one another. *Third,* Shapes are legitimate elements of mathematical reasoning. *Fourth,* The circle is the natural *basis* of all area ; and circumference and radius are the only natural elements of area in all shapes alike. *Fifth,* In geometry, all expressions of numbers are expressions of magnitude. *Sixth,* That geometry is therefore a physical and not a metaphysical science ; and with these general principles Mr. Smith proceeds to demonstrate what the breadth of a line is, which he makes to be always unit, or *one* of the magnitudes in use, whatever the magnitude may be.

Although as I have remarked, Mr. Smith was for more than a year a close student of my papers, and derived all the principles of his reasoning from their examination, and in frequent discussions with him it was explained and vindicated, that the principles which were there set forth were capable of universal application to every part and department of mathematical truth; and although under the circumstances, I might justly have claimed a consultation, yet, I never saw *any portion* of his work, and *knew nothing whatever* of its contents, until it was published. And as Mr. Smith has spoken in his preface of my work as forthcoming, —and has taken up the identical principles which I have developed as the basis of his reasoning, with the acknowledgment, only, that in the demonstration of the Quadrature, I have proved that lines "have breadth," while at the same time he has seen fit in some essential particulars to differ from me in his *conclusions*, it seems to be proper that I should have something to say on the subject. I hold the right therefore at all times to exercise an impartial criticism over his work.

In the *first* place, then, Mr. Smith defines a point to be "just what the books make it," *i.e.*, "position without magnitude." But it will be obvious to any one, I think, that if a point can exist without magnitude, then a line may exist without breadth, and the mathematical definition becomes a positive truth, and there is no necessity for understanding any other line; and if his definition of a point be true, then the basis of his whole argument is at once destroyed.

Secondly. No doubt whatever exists in my mind that the lines which Mr. Smith demonstrates, are the true lines which Nature employs in the *extension* of angular forms,—the truth appears to me to be self-evident, and as I have said that nature is the perfection of geometry, they are of course *the true geometrical lines*, and they are also the true lines which geometers have *always* used, whatever definition they may have chosen to affix to them. But it has been seen in the course of my work, that in practical geometry, lines have other uses *besides the measurement of extension*, and it is self-evident, that Mr. Smith's lines are incompetent to fulfill *all* the uses required, any more than the imaginary lines of geometers without breadth.

It is self-evident also, that Mr. Smith's lines adapt themselves to the properties of *straight* lines, and nothing else; and to those circumstances, and those only, in which it is *entirely immaterial to the result whether* *lines have breadth or not.* They are, therefore, just as good, and no better, than the imaginary lines of geometers without breadth, which adapt themselves only to the measurement of angular forms, and by which mere distance is implied. Nothing, therefore, can be gained, and much inconvenience would be realized from the adoption of *his* proposed change of definition. Notwithstanding, therefore, that I agree with him entirely in respect to the truth of his demonstration, as far as it goes, yet I cannot agree with him in respect to the proposed change, or alter my own expressed opinion in respect to definition; because, I am well satisfied, that no *single* definition is capable of expressing all that is necessary to be understood respecting lines in geometry.

Thirdly. The discovery, (if it can be deemed a discovery,) that lines having breadth, *such* breadth is necessarily *one* at some point of value, is not Mr. Smith's—that point was reasoned and distinctly stated in my papers, which he examined.

Fourth. That the units of which straight lines are composed are cubes, is not Mr. Smith's discovery. It is a necessary consequence of lines having breadth, and right angles being made the standard of quantity. While that is the case, there can be no other unit than a cube; and this point was also distinctly stated by me to Mr. Smith, in discussions upon the subject. The condition, however, applies to lines which are the measure of angular shapes, but *not to curved lines,* which have no angles, either right or otherwise, and which can, therefore, only equal a cube, but can never be made a cube in form, without altogether changing the character of the lines.

Fifth. The whole of Mr. Smith's propositions, from the eighth to the forty-ninth, inclusive, being nearly two-thirds of all that are in the book, are based on ONE *single principle*, or pure *element of geometry.* That one principle is contained in my first proposition, chapter ii., as follows: "All regular shapes formed of straight lines and equal sides have their "areas equal to half the circumference by the *least* radius which the "shape contains (which is always the radius of an inscribed circle), "than which every other radius contained in the shape is greater, and "the circle has *its* area equal to half the circumference by *the* radius to "which every other radius contained in the circle is equal." This proposition Mr. Smith examined in my papers, and there is no other "*element of geometry,*" contained in the whole forty-two propositions above referred to, which is not contained in this ONE. It will be seen at once, that the diameter of a shape being the diameter of an inscribed circle, and its area being half its own circumference by the radius of such inscribed circle, therefore the propositions of Mr. Smith running through the whole catalogue of shapes to prove the general principle, that when circumference equals two, the area equals half the diameter, and when diameter equals two, the area equals half the circumference, &c., &c., &c., are simple truisms of arithmetic based on the *one elementary proposition* above given. They are very well as examples of a method of demonstration, but no element of geometry is contained in any one of them.

Sixth. A large share of those truisms, relating to plane figures, which Mr. Smith calls the "Harmonies of Geometry," are of the same class, and most of them based on the same elementary truth. I have examined them all a hundred times over, before Mr. Smith ever thought of them, explained many of them to *him*, before he began to write his book, and some of those contained in his book are expressed precisely in the language in which he received them from me. They are all harmonies, it is true, and very beautiful ones, too, as every truth of nature is, but they are *not elements*, and I never should have thought of advancing them as such; nor did I ever imagine that they were so new, as to be unknown to every *practical and clear-sighted mathematician.*

If I am justified in these remarks, by the facts, which will speak for themselves, it will be quite apparent that Mr. Smith has misled himself into the belief, that most of his propositions are "new elements" of geometry, which are only simple geometrical truths; and under this self-deception, in respect to their newness and value, he has indulged in a too hasty, and not quite excusable ambition, to be considered the author and discoverer of new principles which should revise and improve the whole of mathematical science.

The method of demonstrating plane and solid figures, by lines having breadth, is one of Mr. Smith's own arrangement, and for which he should receive due credit. It is entirely simple, perfectly conclusive in respect to angular forms, and by me would be adopted in preference to the method of Euclid. His argument that there is but one kind of quantity, and one kind of mathematics, is, I think, sustained, and is, therefore, entitled to a candid, careful and respectful consideration. But the chief merit of Mr. Smith's work is contained, I think, in his fifty-second proposition, as follows: "In all triangles whatever, the whole circumference "bears the same proportion to the base, as the perpendicular of the "triangle bears to the radius of the inscribed circle." This proposition, so far as I know, is entirely new—it is purely elementary—embodies a vast amount of geometrical truth—is clearly and lucidly demonstrated, and, I think, when its capacities are proved, will be found to be of great use; and, therefore, the solution of this problem alone, apart from all others, should be sufficient to entitle Mr. Smith to a high consideration, as a valuable contributor to geometrical science.